THE SCALE PROBLEM FOR OLD MAPS

(and mental maps)

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From the theory of cartography we know that the instantaneous scale at a point on any map is always represented by the equation:

$$dS^2/ds^2 = g_\lambda \cos^2\alpha + 2g_{\varphi\lambda} \cos\alpha \sin\alpha + g_\varphi \sin^2\alpha \,.$$

This shows that the scale on a map is generally different in all directions. One version of the scale problem for old maps is then to estimate the coefficients in this equation at all points and directions of interest.

Given "landmarks" on the map then the earth distances (ds), map distances (dS), and angles (alpha, from an arbitrary reference direction) relating every pair can be measured or, more efficiently, computed. In any event one can identify locations by coordinates, latitude and longitude on the earth and arbitrary Cartesian coordinates on the map, and can present these numbers to a computer program. Then for the i-th measurement at each point the equation which holds is:

$$(dS_i/ds_i)^2 = g_\lambda \cos^2\alpha_i + 2g_{\varphi\lambda} \cos\alpha_i \sin\alpha_i + g_\varphi \sin^2\alpha_i \,.$$

Here we use the map distance (S) as measured on the map (in say, cm) over the ground distance (s, in km). The latter is assumed to be known from a modern map or calculated from known modern coordinates of the landmarks. The map scale to any point is of course map distance over ground distance and is equal to dS/ds.

As long as there are three measurements from each point the coefficients (**g**) can be estimated for that point. Writing the immediately foregoing equations in matrix form, with obvious notation, $\mathbf{S = CG}$, we immediately have the solution $\mathbf{G = C^{-1}S}$.

When more than three distances are measured from a point we use the least squares version $\mathbf{G = (C^tC)^{-1}C^tS}$, or, perhaps using weights $\mathbf{W_i}$ proportional to $\mathbf{exp(-ds_i)}$ or something similar as weights (an alternative is to use only nearby points, i.e. indicator functions as weights), the weighted least squares solution becomes $\mathbf{G = (C^tWC)^{-1}C^tS}$. This weighting seems appropriate since the equation specifies only the instantaneous scale.

Once the **g**'s are estimated the areal and angular distortion at each point can be computed immediately, as can the linear stretch (the scale) in any direction about the point as well as the maximum and minimum scales (Tissot's a, b) . When the map is conformal then $g_\varphi = g_\lambda$ and $g_{\varphi\lambda} = 0$, and for equal area maps one has $g_\varphi g_\lambda = \cos\varphi$. Having estimated the metric coefficients (the **g**'s) one can test to see if either of these relations hold. There will of course also be a constant of proportionality to be estimated for the entire map. This can be taken as the mean of the values obtained above. Regional or local means, and variances, may also be of interest.

http://www.geog.ucsb.edu/people/tobler.htm