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The Geometry of Mental Maps †

ABSTRACT

The empirical estimation of mental maps is advantageously approached as a classical cartographic problem. A geometrical survey can be decomposed into the determination of a configuration, a scale, an orientation, and a positioning with respect to an origin. For each of these the theory of the adjustment of observations leads to the use of the inverse covariance matrix for the weighting of data from several measurements and for specification of the error variances. This theory also provides adjustment procedures that vary depending on whether the data are collected in a graphical mode, or as a traverse, a trilateration, or a triangulation. Furthermore, departures of any configuration from a specified continuous model can be measured using Tissot's theorem. Global measures such as those suggested by Airy, Jordan, and others may also be valuable for measuring overall changes in the perception of configurations over time. Different individuals can be expected to have different perceptions, and this may be interpreted as different viewing points, in the sense of photogrammetry. One is thus lead to inquiries concerning projective invariants, or affine invariants, or other metrical and non-metrical properties of mental maps.

[Denotes equation in oral presentation not in published version]

In modern cartography, one distinguishes between two classes of theory. The first covers the geometrical content of a map and the second treats the substantive content of a map. The geometrical content is extremely well understood. Here the concern is with the *configuration* of control points and the *scale, orientation,* and *location* of this configuration relative to one of the celestial bodies. The theory of the substantive content of geographical maps, on the other hand, is quite unsatisfactory, incomplete, and disjointed - an assertion that I think will be proven by the history of the next several decades. Given this state of the cartographic art, it seems prudent to begin an investigation of mental maps by concentrating on those parts of cartography wherein the achievement of definite results can be anticipated with some certainty, and this is the strategy adopted here. The entire focus is thus on the geometric content of mental maps. It is probably possible to obtain some results via the bivariate modulation transfer function for geographical maps considered as two-dimensional scalar fields, since some theory is beginning to emerge for this restricted class of geographical maps, but this is ignored in the present discussion.

In the classical cartographic paradigm, one knows that the position of any point on the two-dimensional surface of the earth can be established by two independent measurements. In a triangulation one observes angles; in a trilateration one measures distances; and in a traverse one employs a combination of distance and direction measurements. These empirical data are obtained using a variety of instruments, the precise nature of which seems strongly dependent on the available technology. If we wish to estimate a mental configuration that relates terrestrial points to each other, we need to obtain comparable empirical data. But the measuring devices will differ. There are some fairly obvious ways of obtaining data on mental configurations and a few that are more subtle. One can ask people to draw maps, which I would not consider a very good procedure since it confounds drawing ability with geographical knowledge. One can devise questionnaires to elicit estimates of distances or directions, or a combination of the two, between places. One can use a method of paired comparisons, obtaining a rank ordering of distances or directions. One can investigate associations between places (e.g.: given the name of a place, cite another place whose name comes to mind most readily; or, cite a place adjacent to the named place; and so on). Alternately one can infer a configuration from a behavioral pattern. Clearly we are still at the stage where our instruments can be improved. In all of this we assume that the subjects being studied have a representation of their environment and that this is somehow map-like and can be observed by some type of measurement procedure. I am not convinced that the basic assumption is meaningful, but I have been unable to devise an experiment that would force me to give it up. Clearly, some representation of the environment is required, but whether this is hierarchical or map-like is not known. Experiments with computer robots able to move within a complex environment, such as are now under way, may be more illuminating than inferences from biological subjects.

Let us take a cartographic posture and assume that we have obtained, by some empirical procedure, a set of estimated relations between places. If it is reasonable to consider these relations to be distances, then we can use the formula known to every high school student.

$$[\quad D_{ij} = \{(X_i - X_j)^2 + (Y_i - Y_j)^2\}^{1/2} \quad]$$

The mapmaker’s problem is to find the coordinates, given the distances, and not, as is the usual case in high school, the other way around. In two dimensions N points give rise to 2N coordinates, the unknowns, and to N(N-1)/2 symmetric distances, the data, not all of which may have been observed. In general there will be more equations (one for each distance) than unknowns. Each equation can be approximated by a linearization,

$$[\ dD_{ij} = Mf/MX_i\ dX_i + Mf/MX_j\ dX_j + Mf/MY_i\ dY_i + Mf/MY_j\ DY_j + \ldots.]$$

and the entire problem can then be written as a matrix equation

$$[\quad dD_{ij} = (\ Mf/MX_i \ldots.Mf/MY_j)\ (dX_i \ldots.dY_j)^t \quad]$$

Starting from an estimated initial configuration, one iterates on the least-squares solution; and in the cartographic case where the initial guess is a good one, that solution converges rapidly (see [25]). An exactly analogous demonstration is available for relations that can be considered angles. N points taken three at a time yield N(N-1)(N-2)/6 possible measures, assuming symmetry. High school trigonometry again yields directions as a function of the coordinates.

$$[\quad {}_i\alpha_{jk} = \tan^{-1}((Y_j - Y_i)/(X_j - X_i)) - \tan^{-1}((Y_k - Yi)/(X_k - X_i)) \quad]$$

This is linearized using Taylor series and an initial guess for x and y, and the usual solution is again given by iterative least-squares methods.

A variation or combination of the foregoing methods can easily be devised if a combination of distances and angles has been observed. More importantly, in a two-dimensional space of constant curvature three coordinate locations (determining scale, orientation, and position of the configuration) are arbitrary so that if the number of observations, appropriately distributed, exceeds $2N - 3$, then one apparently has excessive information. An axiom of all of the empirical sciences is that every measurement is to some extent wrong. The redundancy of information is thus not only most desirable, it is really necessary. Unquestionably, one of the most important innovations in cartography during the last two centuries has been the development of methods of taking into account the empirical error axiom. We now know that redundant measurements provide a method of evaluating the internal consistency of empirical data. In the cartographic field, the inverse covariance matrix defines the error at each point of the configuration, and this is usually illustrated graphically by the drawing of small error ellipses at each point. It is easy to see how this might be applied to mental maps. Suppose, for example, that a large number of people have estimated the distance (or direction) between points, e.g., between cities in the United States. We can use the mean of the individual distance (or direction) estimates, weighted by the inverses of their standard deviations, to obtain the configuration as a set of points and can also obtain the standard error of the estimate at each point. These standard errors may be more important than the average estimates since they tell us something about the variance of the process. It seems to be a common fault of beginners to present an average mental map without indicating the degree of fuzziness of this map, as measured by the standard error.

The error, it must be stressed strongly, is always relative to some ideal. In the present context the model will always be a geometry. There are of course innumerably many different geometries so that one has no assurance that one's data will not fit some different geometry better than the one currently used as the model. The previous examples solved the triangulation and trilateration problems assuming a plane. Suppose for example, that it is assumed that the mental map or configuration will fit best on the surface of a sphere, of unknown radius. Then the triangulation or trilateration procedure must be modified. On a sphere one only need solve the problem a number of times using a different radius each time. Suppose, to illustrate this, that the maximum observed distance is D. This distance is an upper bound on the length of the arc of some great circle on a sphere of (unknown) radius R. Thus $0 < D < k\pi R$, with $0 < k < 1$. Solve the problem for a sequence of k's, say, from 0.1 to 1.0 in steps of 0.1. At each stage the radius is given by $R = D/k\pi$. A simple solution procedure is to work in a three-dimensional Euclidean space after reducing every distance to a chord distance by a transformation

$$\left[\quad d_{ij} = 2D'/(\pi k) \sin(\pi k d_{ij}/2D') \quad \right]$$

The least-error solution yields an estimate for k, and thus R. The configuration of points is then placed on the sphere by well-known equations, choosing one point and one direction arbitrarily. Alternately the solution can be computed directly on a sphere by using spherical trigonometry. Thus, by assuming that the number of dimensions is two and that the metric is given by the spherical distance formula, we find both the configuration and the curvature of the spherical space. It is a natural generalization for a cartographer to consider spaces that are not flat since solutions to geodetic triangulations on an ellipsoid have been available for more than one hundred years [8]. It is easiest on surfaces of constant curvature because it then does not matter where one is on the surface. This can obviously be generalized to higher-dimensional spaces of constant curvature. For Riemann spaces of variable curvature or for more general Finsler spaces, the problem seems much more difficult [19, 6]. In principle a torus, a piece of Swiss cheese, a potato, or the geometry induced by modern transportation systems, and so on, could be chosen as the geometric ideal to which the adjustment procedure should be modeled. In psychology the multidimensional scaling procedures have been extended to include the Minkowski metric, with the "city-block distance," or Manhattan geometry, as a special case [20]. It also comes as something of a shock to the metrically minded cartographer to learn that one can also determine a set of rectangular coordinates for points, say A through E, when given only the relations of the sort $AE < BC;$ the distance from A to E is less than the distance from B to C. Although no numbers are specified in this exercise, the geometry is quite rigidly determined. Kendall has shown that one only needs to know adjacencies to obtain metrical solutions [12]. The reason is that the redundancies in the geometrical conditions override the lack of quantitative observations. One does not need to have numbers to determine the geometrical configuration of points on a map, and the more observations one has, the less the dependency on numerical values; N(N-1)/2 increases much faster than 2N-3 [21].

It is possible to be somewhat more specific. The Luneberg model, for example, asserts that visual space is three-dimensional Lobachevskian, that is, the model is a pseudosphere [14]. Certain invariances are required of visual objects, and these are said to require the hypothesis of a space of constant curvature. It has similarly been proposed that modern transportation systems induce a two-dimensional metric of constant negative curvature of the surface of the earth. An experiment to test this model for mental maps would require estimates of distances between pairs of places located at varying distances and directions from the observer. The hypothesis is then fairly clear; the estimated distances are a predictable function of the true distances and the distance from the observer. But I would suggest that this hypothesis could also be rejected and that one might find that the experiment would simply show that the errors of distance estimation increase with increasing distance from the observer. That is, the space might become more fuzzy the farther away one goes, and there is not a consistent bias in the distance estimation. The true situation must be more complicated than this because each point in the estimated configuration has about it an uncertainty that is a tensor function of location, not just one number. This uncertainty is generally approximated by the simpler error ellipse. Any satisfactory model would have to describe the spatial pattern of the variation in the size, eccentricity, and orientation of these error ellipses. As a reasonable first approximation, the standard error at each point of the configuration defines a scalar function of position (i.e., a single number), and this is somewhat easier to evaluate than the tensor function.

The result of many of the foregoing estimates and manipulations is a configuration, that is, a set of coordinates for the N points. One may now wish to compare this configuration with some other configuration, a "true" configuration or a configuration obtained for some different group or at a different time period or under alternate experimental conditions, and so on. The similarity between two mental maps, or between a mental map and some standard, is to be measured. To keep the exposition simple, it is convenient to consider only the two-dimensional Euclidean case, but this does not detract from the generality of the argument. Comparable methods are easily available for any model one chooses to specify.

If we can identify corresponding entities in the two images, we can construct a table of correspondences. One set represents coordinates of the j^{th} point in one image, and the other set represents the coordinates of the associated point in the other image. We now assume that there exists a mapping between the sets,

$$[\quad (X,Y) \rightarrow (U,V) \quad]$$

and we wish to investigate the properties of this transformation. We can estimate the relation by noting that there is in this a formal similarity to the problem of producing two superimposed contour maps from observational data on scattered locations. We can thus use standard contour-drawing computer programs to create empirically a picture of the functions as a transformation of a coordinate grid. Two recent dissertations at the University of Michigan use this technique with empirical data from Toledo (United States) and from Cologne (Germany). The method has long been used in cartography to study ancient maps [24]. To obtain the distorted grid, we must use an interpolation procedure to go from the isolated observations to a field of data, in effect invoking two assumptions. The first assumption is that the functions are effectively, at least piecewise continuous and everywhere defined. It is also convenient to assume that they are, one-to-one and single valued. The conventional wisdom seems to admit to these types of assumptions. Lynch [15], for example, reports that informants draw rubber-sheet maps of cities. He does not actually provide the evidence but comments on informant drawn maps as follows:

> There was a strong element of topological invariance with respect to reality. It was as if the map were drawn on an infinitely flexible rubber sheet; directions were twisted, distances stretched or compressed, large forms so changed from their accurate scale projection as to be at first unrecognizable. But the sequence was usually correct, the map was rarely torn and sewn back together in another order. This continuity is necessary if the image is to be of any value. (p. 87)

The second assumption required for the interpolation procedure is that one can predict the values of the function at unobserved locations from the values at observed locations. As has been outlined most clearly by Heiskanen and Moritz , based on Wiener's theory, the statistical validity of such interpolation (really a prediction procedure) depends on a knowledge of the two-dimensional autocovariance function of the data, and this is equivalent to theoretical knowledge of the process that produced the observations. The apparent efficacy of the double-contour interpolation procedure is therefore somewhat misleading and spurious, or contains implications concerning the process governing the generation of the observations. This circularity should not be given too much weight; illustration of the transformation using the deformed grids appears useful. Naturally, since all objects, for example in a city, can be located with respect to latitude and longitude coordinates, the interpolation procedure can very easily be extended to draw complete deformed street maps, not only the distorted grid and not only those items cited by informants. Finite-difference methods can also be used to obtain the partial derivatives of the functions from the deformed grid, and these can be used to investigate the type and amount of distortion, as described below.

An alternate to the foregoing empirical-graphical contouring procedure is to postulate an explicit form for the relation between the sets. It is too much to expect that the empirical data will give an exact fit to the postulated function so that it is appropriate to use least-squares methods. Thus, it is desired to find the numerical values of the parameters such that the appropriate residual

$$\left[\ \sum_{j=1}^{n} \{(U'_j - Uj)^2 + (V'_j - V_j)^2\}\ \right]$$

is minimized. We still need to choose a specific model. The simplest postulate is the two-dimensional equivalent to a linear regression line, namely, a Euclidean transformation [23]

$$\left[\; U' = a_{11} X + a_{12} Y + a_{13} + \varepsilon \;\right]$$
$$\left[\; V' = a_{21} X + a_{22} Y + a_{23} + \varepsilon \;\right]$$

This transformation is insensitive to rotations, translations, and changes of scale, but calculates how large these are. It measures only the association between the two configurations. Using this model it is easy to compute a quantity exactly equivalent to the usual Pearsonian correlation coefficient, as the ratio of the regression variance to the total variance. One such computation, using data from Toledo, yielded a fit of circa 80 percent between a configuration calculated from questionnaires and the usual city map .

To a cartographer, a natural, literal interpretation of the statement “everyone has his own point of view” would be to assume a viewing point in space, which might thus result in a representation similar to a low oblique aerial photograph with the area near the viewer enlarged relative to distant locations. Mathematically this is the statement

$$u = \frac{a_{11}x + a_{12}y + a_{13}}{a_{31}x + a_{32}y + a_{23}}$$

$$v = \frac{a_{21}x + a_{22}y + a_{22}}{a_{31}x + a_{32}y + a_{23}}$$

as is well known. The solution requires that the coordinates of the viewing point in space be found when given the coordinates of corresponding points in the original and the image. This is the space resection problem of the photogrammetrist, and solution algorithms are readily available [4, pp. 36-44]. Thus, although the hypothesis is a bit implausible, it is easy to test and provides an alternate to the interesting joint-space solution illustrated by Lieber [13]. The projective hypothesis also has other interesting implications. Under a projective transformation all straight lines remain straight, and the cross-ratio is an invariant. One might design an investigation specifically focused on these invariants, and this strategy can be carried over to other postulated transformations. This would be F. Klein’s suggestion, investigating invariants along a hierarchy of geometries. It would be, for example, very interesting to have a measure of the magnitude of a departure from one-to-oneness or of the magnitude of departures from differentiability or from continuity, thus explicitly measuring the amount of agreement with Lynch’s statement. I am not aware of proposals for any such measures.

Other possible specific transformation rules come to mind rather readily, of course. One such is the general least-squares complex polynomial, which is mathematically very simple. Requiring this solution to be analytic, of course, specifies that the transformation be conformal. From a substantive point of view, a conformal transformation might be justified on the grounds that it requires that the transformation, in the immediate vicinity of any point, be a simple change of scale. This property of conformal transformations can be visualized by imagining that one is looking at the original surface through a microscope mounted on wheels and constructed in such a way that any movement over the viewing surface changes the amount of magnification. At any one time it is possible to see only a small portion of the transformed original through the eyepiece of the microscope, and each small portion will be seen at a different magnification but is otherwise identical to the original. In the present context the surface under scrutiny might be the area of a city, seen in larger magnification in familiar parts. Kabrisky [11], going beyond the evidence, puts forth the related suggestion that the visual field is conformally represented on the cortex of the mammalian brain. I am not aware of any physiological evidence for a comparable representation or localization of geographical information in the brain. And there are arguments against conformality, in spite of its apparent plausibility. The very mathematical simplicity that renders conformality an appealing hypothesis also severely circumscribes the possible transformations. The more conservative scientific approach would be to let the data speak for themselves, that is, design the experiment to explicitly test the hypothesis of conformality. The cartographic point of view also suggests that consideration be given to equal area mappings and to quasi-conformal cartograms. For this we need Tissot's results on the distortion of two-dimensional transformations, assuming a two space with a Riemann metric [22] .

Given a differentiable transformation, the ratio of the distance dS at a location in the u, v image to the corresponding distance ds in the x, y original is given by

$$\frac{dS}{ds} = \left[g_x{}^2 \cos^2 \alpha + 2 g_{xy} \sin \alpha \cos \alpha + g_y{}^2 \sin^2 \alpha \right]^{1/2}$$

where the g's are particular combinations of the partial derivatives u_x, u_y, v_x, v_y. These quantities may be obtained by formal differentiation of the functional equations or from finite-difference graphical methods from the empirical contour nets, perhaps after smoothing. The distance ratio defines the scale at any particular position on a map. The thing to notice is that the distance ratio depends on an angle α. There is no such thing as the scale at a particular position on a map, but rather at each location there are an infinite number of scales, one in each direction. The scale equation, of course, represents an ellipse, Tissot's indicatrix, and has a major and minor axis, as do all ellipses. These represent the major and minor scale axes. If they are of the same magnitude, the ellipse becomes a circle, and the transformation is conformal. The areal exaggeration is defined as the product of these axes. The maximum angular distortion at the location in question is given by a trigonometric function of the scale axes. There are thus several well-defined quantities that can be used for the evaluation of mental maps. In order to obtain these quantities, it is necessary to compute the partial derivatives, and we can establish hypotheses based on these derived quantities. For example, it is now possible to postulate that the areal exaggeration of a mental map is proportional to knowledge that one has of the location in question. One might ask informants to estimate distances (or to draw a map, etc.) between places (e.g., stores) with which they are familiar in a city. Then one would expect each person's actual knowledge of an area to be proportional to the product of the travel propensity, t(U,V), and the distribution of activities, p(X,Y), and the total knowledge of all persons to be the sum over the entire area of all of these products

$$\left[\sum_{\text{-inf}}^{\text{+inf}} K(X,Y) = \int\int_{\text{-inf}}^{\text{+inf}} K(X,Y)dudv = \int\int p(X,Y)\ t(X-U), (Y-V)dudv \right]$$

The result of course is a convolution integral in two variables. This then is the expected distribution of places cited for a large random sample of informants. Models similar to this have been proposed for specific urban population distributions by Moore [17], Moore and Brown [18], and Dacey [5], and they have also included estimates of the expected spatial arrangement of the variances. For the usual models of city population densities and of spatial interaction fields, the maximum of k(x, y) is expected to lie between the residential location of the individual and the center of the city. A specific hypothesis now might be that the standard error of the positioning of any place (as computed from many observations pertaining to the relative location of that place) should be related to the lack of knowledge concerning that place. That is, the standard error of the estimate should be inversely proportional to how well the area is known. This seems a rather trite statement, but certainly testable. The alternate hypothesis, that the mental image is proportional to the amount of knowledge, either for an individual or for a number of individuals, is now also stated in a testable manner. And this might equally apply to cities, to an entire county, or to the entire world. The convolution integral and the areal exaggeration of the mental image are definite - albeit unknown - numbers and are directly comparable. In principle they are capable of being measured independently of each other. Given an expectation (the convolution integral) one may be able to calculate an expected mental map based on such hypotheses.

One can also work with the distances. Suppose one has, for N places on the earth, an estimate of their separation as judged by people. This will yield N(N-1)/2 distance estimates. Each place will be connected to N - 1 places, and one can form the ratio dS/ds for each pair. Assuming that the locations are known, it is also possible to compute the directions relative to some arbitrary initial reference. We can then write the N-1 equations and, as long as N-1 exceeds three, can form the least-squares estimate of the coefficients (in matrix form)

$$[\quad (dS/ds)_i^2 = (\cos^2\alpha_i \;\; \sin\alpha_i \cos\alpha_i \;\; \sin^2\alpha_i)\,(g^2{}_x \;\; 2g_{xy} \;\; g^2{}_y)^t \quad]$$

The distance ratios, however, refer to finite and not infinitesimal values so that it may be more appropriate to weight them by a function, normalized so that the smallest distance in the set has largest weight. The least-squares solution is then the usual weighted case. The g's here can be related to Gauss' first fundamental form, from which the curvature, and thus the implicit geometry, might be calculated. This process can then be repeated for each of the N locations. When the curvature is known at every point, the geometry is known. Of course, we have only a finite set and not a continuum, which must be postulated.

Other measures of the distortion at a single location have been used in the theory of map projections. Generally they describe a combination of angular, areal, and distance distortion. Included here would be Jordan's measure [10], the Jordan-Kavraisky measure, the mean scale ratio, or the several measures proposed by Airy [10, 1]. A detailed discussion of these various quantities has been given by Biernacki [3] and by Meshcheryakov [16].

Global measures are then obtained by taking either the supremum of one of the foregoing or by the average value taken over all locations. One can thus obtain several overall measures of the degree of correspondence of spatial configurations. Each measure is slightly different, but they all tend to be monotonically related so that it usually does not matter which measure is employed, unless of course there are compelling reasons for a particular choice. As an example, the total error might be defined for a mental map of the world by

$$\frac{\int_{-\pi}^{\pi}\int_{0}^{2\pi}\int_{0}^{2\pi}\left(\frac{dS}{ds}-1\right)^{2}\cos\varphi\, d\alpha\, d\lambda\, d\varphi}{2\pi\int_{-\pi}^{+\pi 2}\int_{0}^{\pi}\cos\varphi\; d\lambda\; d\varphi},$$

but there are several other possibilities.

In summary it is postulated that one can elicit information concerning locational configurations from people and that these configurations can be embedded in a two-dimensional continuum which is sufficiently like conventional maps that they can be compared. Approached conservatively as a classical cartographic problem, the hypotheses are sufficient to allow the derivation of a number of potentially useful measures. The theory of the adjustment of observations leads to the use of the inverse covariance matrix for the weighting of data from several measurements and for the specification of the error variances. This theory also provides adjustment procedures that vary depending on whether the data are collected in a graphical mode or as a traverse, a trilateration, or a triangulation. Furthermore, departures of any configuration from a specified continuous model can be measured using Tissot's theorem. Global measures such as those suggested by Airy, Jordan, and others may also be valuable for measuring overall changes in the perception of configurations over time. Different individuals can be expected to have different perceptions, and this may be interpreted as different viewing points, in the sense of photogrammetry. One is thus led to inquiries concerning projective invariants, affine invariants, or other metrical and nonmetrical properties of the geometry of mental maps.

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