Numerical Approaches to Map Projections +

Waldo R. Tobler, Michigan, USA

With 3 figures and 6 tables

Abstract: The classical approach to map projections is to obtain an explicit pair of equations relating places on the earth to locations on a map. Modern computational facilities render feasible an alternate approach using numerical analysis techniques. Surveying adjustment procedures such as triangulation and trilateration constitute examples familiar throughout the cartographic community. In this instance the principle of least squares can be used to define map projections with useful properties. To take advantage of these approaches one employs further numerical methods, including bivariate interpolation and splining, and finite difference approximations to the derivatives of transformations. The point of view described is especially valuable when presented with a map whose projection equations are not known explicitly.

LAMBERT (1) demonstrated that an appropriate derivation of a map projection begins with a definition of the properties to be preserved. This definition typically takes the form of a partial differential equation that covers the class of map projections in question. Particular solutions are then obtained by specifying additional conditions. To take a simple case, a conical equal area projection of a sphere is usually derived in the following manner. The equal area condition, in geographical and plane polar coordinates, is $r\,(\partial r/\partial\varphi \;\; \partial\theta/\partial\lambda - \;\partial r/\partial\lambda \;\; \partial\theta/\partial\varphi\,) = R^2 \cos\varphi$ and conical projections, by definition, are given by equations of the form $r = f(\varphi)$, $\theta = n\lambda$, where $0 < n < 1$ is known as the constant of the cone. Clearly $\partial r/\partial\lambda = 0$, $\partial\theta/\partial\varphi = 0$, $\partial\theta/\partial\lambda = n$, so that $r\,\partial r/\partial\varphi\; n = R^2\cos\varphi$. Solving for r yields $r^2 = 2\,R^2/n \sin\varphi + C$ which contains two arbitrary constants; the constant of the cone, n, and a constant of integration, C. The usual boundary condition is to make the length of two parallels on the map equal to their length on the sphere, or minor variants of this condition. This completely determines the constants n and C. For example, let φ_1 and φ_2 be the two standard parallels. The length of any parallel on the sphere is $R \cos\varphi\, d\lambda$. The length of the parallel on the map is $r\, n\, d\lambda$. Set these equal to each other and solve for n. In the present instance there are two standard parallels whose images are arcs of circles of radius r_1 and r_2. Each yields an equation for n, but n is a constant so that these equations must equal each other Some simple algebra now yields $2n = \sin\varphi_1 + \sin\varphi_2$. With this value of n, C is determined by $C = R^2\,(1 + \sin\varphi_1 \sin\varphi_2)/n^2$. For example if $\varphi_1 = 30°$ and $\varphi_2 = 45°$, then $n = 0.604$ and $C = 3.72$. This is ALBERS' equal area conic projection with two standard parallels.

Least squares methods might also be used to determine the constants This particular choice - the method of least squares - combines several advantages, of which historical continuity, wide familiarity throughout the scientific community and the relation to Gaussian probability statistics weigh heavily. Other criteria are easily formulated but yield results that differ only in detail not in overall strategy. The CHEBYSCHEV criterion as used in the example below is not further discussed here. The application of least squares methods to cartography is of course not new. For example, a suggestion in 1861 by G. AIRY (2) was to balance departures from conformality against departures from equivalence. If a is the maximum, and b the minimum, value of the linear exaggeration, then the "balance of errors" criterion can be stated as minimization of $\iint (a/b - 1)^2 + (ab-1)^2 \, dA$, where the integration is over the entire region. The expressions for a and b for an arbitrary map projection are complicated, and no general solution has been found to this difficult problem. YOUNG (3) made an intensive study of the subject, and relevant literature has recently been summarized by MESCHERYAKOV (4) and by BIERNACKI (5). For an equal area conical projection ab = 1 and $a/b = \cos^2 \varphi/(n^2C - 2n \sin \varphi)$ so that AIRY'S integral is considerably simplified. The proposed solution would then appear to minimize departure from conformality.

The modern high speed digital computer also permits one to approach the problem using numerical calculations and finite measures in place of differentials of distortion. A tedious, but direct, calculation over the region of interest, varying n and C, now appears feasible. For example Figure 1 shows a graph of the difference between spherical distances and distances on

Table I. Errors as a function of the two available constants

	c = 3.50					C					c = 4.00
n = 0.575	2383	2106	1837	1578	1327	1082	0843	0611	0400	0255	0224
	2151	1876	1611	1354	1104	0861	0624	0405	0251	0220	0408
	1921	1650	1388	1133	0884	0642	0413	0250	0214	0406	0624
	1696	1428	1167	0913	0666	0430	0252	0206	0399	0621	0839
	1474	1207	0948	0696	0452	0259	0198	0387	0612	0834	1054
n	1255	0990	0732	0482	0273	0189	0369	0600	0825	1048	1268
	1037	0774	0518	0296	0133	0346	0580	0811	1037	1261	1481
	0822	0561	0325	0182	0318	0556	0791	1022	1249	1473	1693
	0610	0363	0192	0284	0527	0766	1000	1232	1461	1686	1908
	0409	0208	0245	0492	0735	0945	1210	1442	1671	1898	2121
n = 0.625	0239	0208	0451	0698	0943	1183	1419	1652	1882	2110	2334

ALBERS' equal area conical projection with two standard parallels. The absolute value of the difference is plotted as a function of the two available constants n and C. The numerical values on which the graph is based are given in Table I. Each computation compares 595 spherical distances with 595 map distances using the points defined by latitude = 22.5 (7.5) 52.5 degrees; longitude = -125 (10) -65 degrees. The constant of the cone takes on the values 0.575 (0.05) 0.625 and C ranges over 3.50 (0.05) 4.00. The minimum of the function $\varepsilon = f(n, C)$, based on the 4,283,702 distance comparisons, occurs at n = 0.610, C = 3.65, and this specifies these values for an equal area conical projection of the United States on which distance errors are as small as possible. It should be emphasized that this solution uses finite measures, not differentials of distortion, and numerical calculations quickly and easily performed on a digital computer. The resulting constants do not differ appreciably from those obtained by more traditional methods, but this example was only for expository purposes. The topography of the error space is clearly exhibited here as a shallow valley between n and C, and which appears as though it might well be approximated by a hyperbolic parabaloid, and in which a gradient-following mathematical mountaineer could easily find the minimum. Any projection that contains arbitrary constants invites exploration by numerical expeditions of this kind.

Minimization of distance errors, when it is known that isometric mappings of a sphere onto a plane are not possible, has numerous applications. As a particular example, consider a case in which one has available only information obtained from a travel itinerary (distances between points, but not the latitudes and longitudes of the points) and one still wishes to construct a map. One may imagine that the ancients might have used such information for the construction of medieval maps. Ruler and compass approximations can be devised with relative ease but it is more useful to formulate a general mathematical solution, as described in the following paragraphs.

To begin it is assumed that one knows the distance between all pairs of points. If there are n points this requires n (n - 1) / 2 distances. The problem is now to determine the coordinates of the n points. The answer can be checked by comparing the original distances with distances calculated from the coordinates obtained through the analysis. The first step is to array the distances in an n by n matrix so that any entry, D_{ij}, is the distance from the i^{th} point to the j^{th} point, as in a road atlas distance table. This matrix will be symmetric since the original distances are symmetric. Then form a new n by n matrix with elements obtained from the distances as follows. The components of the eigenvectors corresponding to the two largest eigenvalues of this d^*_{ij} matrix are the desired map coordinates, with origin at the centroid of the distribution of points and with axes corresponding to the principal axes of the distribution of points. A computer

$$d^*_{ij} = \frac{1}{2}\left\{- D^2_{ij} + \frac{1}{n}\sum_{i=1}^{n} D^2_{ij} + \frac{1}{n}\sum_{j=1}^{n} D^2_{ij} - \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} D^2_{ij}\right\}.$$

program to perform these calculations has been published (6).

The foregoing result is based on theorems described in the psychometrical literature (7):

a) If a matrix d^*_{ij} is positive semi-definite (i. e., has non-negative latent roots), then the distances can be considered to relate points lying in a real Euclidean space.
b) The number of positive latent roots is equal to the dimensionality of the set of points.
c) If the rank of an n by n positive semi-definite matrix is $r < n$, then the matrix can be decomposed into the product of an n by r rectangular matrix of r characteristic vectors and its r by n transpose. The n elements of the characteristic vectors can be considered the projections of the points on r orthogonal axes in the r-dimensional real Euclidean space.

The application in psychology is to locate stimuli relative to each other when given only their dissimilarities (distances). In the present instance the objective is to produce a geographical map and both the Euclidean metric and a restriction to two dimensions are appropriate. Thus only the first two eigenvectors are considered. Since the original distances may not have come from a two dimensional Euclidean space, or may contain measurement errors, the method cannot always give a perfect fit. A measure of the adequacy of the fit is of course available by comparison of the original and resulting distances, and this is also a measure of the adequacy of Euclidean geometry as a description of the space.

The procedure has been tested by computing 2,080 spherical distances between a set of 65 regularly spaced latitude and longitude intersections covering the United States. From these distances the 65 by 65 scalar product matrix d^*_{ij} was formed and the eigenvectors corresponding to the two largest eigenvalues were obtained. The map distances between the points were computed by considering the components of the eigenvectors to correspond to plane map projection coordinates. The mean scale ratio of the resulting distances to the original distances was then used to obtain a multipler (0.99502) to be applied to the coordinates for a final scale adjustment. A small selection of the original and resulting distances are given for comparison in Table II.

Table II. Original and resulting distances

	1	2	3	4	5	6	7	8	9
1	00.000	27.669	55.025	15.000	29.804	53.120	30.000	37.751	54.210
2	27.844	00.000	27.669	29.804	15.000	29.S04	37.751	30.000	37.751
3	55.081	27.844	00.000	53.170	29.804	15.000	54.210	37.751	30.000
4	15.350	29.947	52.910	00.000	23.698	46.741	15.000	25.681	43.577
5	29.608	14~970	29.608	23.594	00.000	23.698	25.681	15.000	25.681
6	52.910	29.947	15.350	46.548	23.594	00.000	43.577	25.681	15.000
7	30.594	37.704	53.982	15.265	25.555	43.642	00.000	18.130	35.442
8	37.710	29.860	37.710	25.749	14.890	25.749	18.256	00.000	18.130
9	53.982	37.704	30.594	43.642	25.555	15.265	35.912	18.256	00.000

Values in degrees. Original distances above the diagonal. The selection of points is (1) 22.5N, 125W; (2) 22.5N, 95W (3) 22.5N, 65W (4) 37.5N, 125W (5) 37.5N, 95W (6) 37.5N, 65W; (7) 52.5N, 125W; (8) 52.SN, 95W; (9) 52.5N, 65W. The regression is: Resulting Distance = 0.29 + 0.99 * Original Distance. R^2 =1.00, N= 36

To see the relation to least squares methods the problem can be reformulated to find a solution which minimizes the sum of the squares of the differences between the original distances and the resulting map distances, i. e., minimize

$$\varepsilon = \sum_k (D_{ij} - d_{ij})^2 \text{, where } d_{ij} = [(x_i - x_j)^2 + (y_i - y_j)^2]^{1/2}.$$

Readers with a geodetic background will recognize this as the method used to adjust measurements obtained in a trilateration survey (8). The function d is linearized using the TAYLOR series

$$d_{ij} \approx \mathring{d}_{ij} + \frac{\mathring{x}_i - \mathring{x}_j}{\mathring{d}_{ij}} dx_i + \frac{\mathring{y}_i - \mathring{y}_j}{\mathring{d}_{ij}} dy_i + \frac{\mathring{x}_j - \mathring{x}_i}{\mathring{d}_{ij}} dx_j + \frac{\mathring{y}_j - \mathring{y}_i}{\mathring{d}_{ij}} dy_j,$$

where the superscript denotes an initial (arbitrary) guess at the solution. Substitution of this equation into $\sum_k (D_{ij} - d_{ij})^2$ allows for the small corrections dx, dy by least squares methods. This leads to an improved guess at the coordinates. The improved guess is then used to repeat the process. Iteration ceases when the coordinates no longer change. This formulation makes clear the procedure to be used when some of the n (n - 1) / 2 observations are lacking, or when they are of unequal accuracy. In the latter case one uses a weighted least squares procedure, with inverse variances as weights. But one could also weight distances in some, presumably important, part of the map more heavily than others, or could choose weights to be some function of the distances themselves; a negative exponential for example would lay the stress on the preservation of local distances. Also suggested is how the same method might be employed to minimize angular distortions.

To obtain map projections which are "nearly conformal in the large" the problem is posed as follows: Given m angles ${}_i\theta_{jk}$ measured from j to k at the i^{th} point, find positions identified by coordinates such that the angles ${}_i\alpha_{jk}$ calculated from these coordinates are as nearly as possible the same as the given angles. The condition to be minimized, with angles in radian measure, is $\sum_m ({}_i\theta_{jk} - {}_i\alpha_{jk})^2$, where

$$
{}_i\alpha_{jk} = \tan^{-1}\left(\frac{y_j - y_i}{x_j - x_i}\right) - \tan^{-1}\left(\frac{y_k - y_i}{x_k - x_i}\right).
$$

This is linearized using TAYLOR series and an initial guess for x_i, x_j, x_k, and y_i, y_j, y_k as

$$
{}_i\alpha_{jk} \approx {}_i\alpha^{\circ}_{jk} + \left[\frac{y^{\circ}_j - y^{\circ}_i}{(d^{\circ}_{ij})^2} - \frac{y^{\circ}_k - y^{\circ}_i}{(d^{\circ}_{ik})^2}\right] dx_i + \left[\frac{x^{\circ}_i - x^{\circ}_j}{(d^{\circ}_{ij})^2} - \frac{x^{\circ}_i - x^{\circ}_k}{(d^{\circ}_{ik})^2}\right] dy_i +
$$

$$
+ \left[\frac{y^{\circ}_i - y^{\circ}_j}{(d^{\circ}_{ij})^2}\right] dx_j + \left[\frac{x^{\circ}_j - x^{\circ}_i}{(d^{\circ}_{ij})^2}\right] dy_j + \left[\frac{y^{\circ}_k - y^{\circ}_i}{(d^{\circ}_{ik})^2}\right] dx_k + \left[\frac{x^{\circ}_i - x^{\circ}_k}{(d^{\circ}_{ik})^2}\right] dy_k.
$$

The result is of course comparable to the solution of geodetic triangles by the method of variation of coordinates. Iteration of the least squares corrections to the coordinates converges rapidly in these cases. In the present context it is suggested for the derivation of map projections, so that ${}_i\theta_{jk}$ might be angles on a sphere or ellipsoid.

In the case of distances, with n points, there are n (n - 1) / 2 possible symmetric distances (n things taken two at a time) and 2n coordinates to be found. Three coordinates are arbitrary, so that a least squares solution is possible only if the number of distances exceeds 2n - 3. For directions, n things taken three at a time yields n(n - 1)(n - 2)/6 angles, and three coordinates can again be chosen arbitrarily, so that at least $n(n-1)(n-2)/6 > 2n - 3$ angles must be known for a least squares solution to exist. The analogy to classical surveying also suggests use of the classical error measures; standard error ellipses at each point (not to be confused with Tissot's indicatrix), and the root mean square error as an overall measure of the goodness of fit. Since the surveyor's mean square error is essentially equivalent to the statistician's variance, it becomes obvious that one can treat a map projection as an hypothesis relating to the geometry of the earth, and that one can compute the percentage of the total variance accounted for by the model. Thus the reader may use Table II as an example from which to compute an estimated correlation coefficient (R^2) from which he can decide whether or not to accept the procedure proposed. All of these new suggestions for the construction of map projections are quite general. There is no requirement, for example, that the distances be symmetrical, and with appropriate modifications (9), the methods can be applied to information measured on an ordinal, rather than ratio, scale. Thus "far, farther, farthest" may contain enough information to compute coordinates for a map (10). The initial distances could also be travel costs, travel times, or road distances (11), instead of spherical distances, or the number of employment opportunities between places i and j.

Table III. Empirical map projection coordinates for a map of the United States

Longitude		95W	90W	85W	80W	75W	70W	65W
Latitude								
52.5 N.	x =	0.0	3.063	6.115	9.142	12.133	15.075	27.956
	y =	13.440	13.531	13.806	14.264	14.905	15.728	16.734
45.0N	x =	0.0	3.527	7.043	10.535	13.991	17.398	20.745
	y=	6.006	6.107	6.409	6.912	7.618	8.526	9.637
37.5N	x =	0.0	3.946	7.880	11.792	15.670	19.501	23.274
	y =	-1.450	-1.343	-1.021	-0.485	0.268	1.237	2.425
30.0N	x =	0.0	4.318	8.626	12.913	17.169	21.382	25.541
	y =	-8.926	-8.815	-8.481	-7.924	-7.142	-6.134	-4.896
22.5 N	x =	0.0	4.645	9.280	13.897	18.487	23.038	27.540
	y =	-16.420	-16.308	15.969	-15.403	-14.608	-13.582	12.320

Values in degrees. Western half to be obtained by symmetry.

As finite procedures the foregoing methods do not result in formulae of the form x = f (φ, λ), y = g (φ, λ) as one would obtain from the more classical approach. Instead, the definition of the map projection now consists of a table of coordinates, and not a pair of equations. Table III gives the result for the 65 latitude, longitude points cited earlier. Of course this table specifies a rule associating places on the sphere with places on the map, just as would a pair of equations. And the table can be expanded to arbitrary density by interpolation. Thus the x, y coordinates corresponding to a point at 82° W, 35° N would be x = 10.479, y = -3.182 by linear interpolation, and would be x = 10.567, y = -3.204 by bivariate splining (12), which gives a slightly smoother result. Neither method of interpolation is very complicated. In order to obtain a complete map the interpolation procedure is applied to all of the geographical data of interest, which might, for example, be stored on magnetic tape. Figure II demonstrates such results. In this instance third order bivariate interpolation was applied, using table III, to each of 10,408 latitude/longitude points used to define an outline map of the United States. The computation and plotting take about one minute in today's computer environment.

Figure III shows a map of the Mediterranean prepared as was the map of the United States with the difference that the distances have been computed along spherical loxodromes and not along great circles. The hypothesis is that this is a reasonable analogy to the construction of Portolan Charts in the 13th century, but the literature is controversial.

Table IV defines another map projection, an equal area pseudo-cylindrical known as the hyperelliptical projection (13). In this case the equations are known,

$$x = \left\{ \alpha + \frac{1-\alpha}{\gamma} \left(\gamma^k - y^k \right)^{1/k} \right\} \lambda$$

$$\sin \varphi - \alpha y + \frac{\alpha - 1}{\gamma} \int_0^y \left(\gamma^k - y^k \right)^{1/k} dy = 0,$$

where α, k, and γ are constant parameters. Inspection of the equations reveals that the y coordinate is defined implicitly, and thus that it is necessary to use an iterative numerical technique to obtain the value at any point. In this instance it may be preferable to perform the exact computations once and then to let Table IV define the projection thereafter, using interpolation from this table for further (arbitrary) points, rather than the more complicated iterative solution. MOLLWEIDE's projection, a special case of the hyperelliptical, provides an example which has been in the textbooks in this form for many years.

Table IV. Table for the construction of a 10 by 20 world map on the HYPERELLIPTICAL EQUAL AREA PROJECTION

Latitude	Y Coordinate	X Factor*
0	0.000000	0.055555
5	0.367979	0.055429
10	0.734807	0.055242
15	1.096182	0.054973
20	1.454780	0.054595
25	1.802264	0.054091
30	2.144195	0.053442
35	2.475016	0.052627
40	2.794726	0.051625
45	3.103326	0.050406
50	3.398039	0.048940
55	3.678861	0.047181
60	3.940243	0.045070
65	4.184957	0.042524
70	4.410173	0.039413
75	4.610387	0.035519
80	4.782840	0.030411
85	4.921966	0.022944
90	5.000000	0.000000

*Multiply longitude by this amount to obtain the X coordinate. The table is computed from the equations given by TOBLER, 1973, using alpha = 0, K = 2.5, gamma = 1.1831.

Table V. Tissot's Measures of Distortion.
Calculated for an Empirical Map Projection of the USA

Lat.	Lon.	A	B	S	2W	K	H	Ω
30.00	-70.00	1.02404	.99235	1.01620	1.8013	1.00090	1.01568	91.5932
30.00	-75.00	1.01609	.99268	1.00865	1.3352	.99952	1.00936	91.2115
30.00	-80.00	1.00962	.99321	1.00277	.9386	.99850	1.00439	90.8762
30.00	-85.00	1.00452	.99414	.99863	.5952	.99778	1.00089	90.5678
30.00	-90.00	1.00058	.99553	.99612	.2900	.99737	.99875	90.2789
30.00	-95.00	.99800	.99720	.99521	.0459	.99720	.99800	90.0000
37.50	-70.00	1.01446	.99457	1.00895	1.1344	.99628	1.01278	90.6333
37.50	-75.00	1.00777	.99442	1.00214	.7640	.99565	1.00655	90.4408
37.50	-80.00	1.00256	.99439	.99693	.4689	.99526	1.00169	90.2893
37.50	-85.00	.99883	.99430	.99313	.2604	.99486	.99826	90.1723
37.50	-90.00	.99648	.99444	.99094	.1178	.99472	.99620	90.0818
37.50	-95.00	.99547	.99477	.99026	.0403	.99477	.99547	90.0000
45.00	-70.00	1.01076	.99628	1.00700	.8271	.99693	1.01012	89.6589
45.00	-75.00	1.00505	.99588	1.00091	.5256	.99706	1.00389	89.6490
45.00	-80.00	1.00105	.99532	.99636	.3287	.99736	.99901	89.6852
45.00	-85.00	.99884	.99420	.99305	.2663	.99760	.99545	89.7643
45.00	-90.00	.99791	.99310	.99102	.2770	.99766	.99335	89.8775
45.00	-95.00	.99759	.99267	.99027	.2834	.99759	.99267	90.0000

Values for the western portion of the map can be obtained by symmetry.

The two tables, III and IV, thus illustrate two alternate numerical approaches to the study of map projections. The projection given by table IV was derived in the conventional fashion, whereas that of Table III was derived from finite calculations. One assertion would be that the map illustrated in Figure II is *prima facie* evidence for the reasonableness of the empirical derivation used to obtain this projection. But it is not necessary to rely on such visual evidence. It is a simple matter to approximate partial derivatives, by standard numerical techniques, from such tables of coordinates. One needs these to describe the distortion on a map using the theorems of Tissot (14). The results of such a computation are shown in Table V. One sees that the distortions are generally less than two percent, and thus that this empirical map projection compares favorably with ALBERS' or LAMBERT's conical projections with two standard parallels. But the empirical projection is not a true conic, since there is a slight curvature to the meridians.

Table VI presents an even more interesting empirical situation. The x and y coordinates represent physical measurements made on an old map. The latitudes and longitudes are the modern spherical coordinates for a sample of identified locations on the so-called Gough map, dating from the fourteenth century (13). Now this table is exactly the same as the previous tables, except for two circumstances. The first difference, of minor import, is that the table is not given at equal increments of latitude and longitude, nor equal increments of x and y**.** If one wished to draw a map from the projection defined by Table VI then the interpolation procedure is more complicated because of this irregular arrangement of the observations. The computation of the partial derivatives, which must be used to analyze the distortion of any projection (16), after the manner of Tissot, is also more complicated. But both can be done (17). The major difference between Table VI and Tables III and IV, however, is in its manner of derivation.

Table VI. Spherical and Plane Coordinates for a selection of Points on the Gough Map

Latitude*	Longitude*	X Coordinate**	Y Coordinate**
58.43	—3.12	6.7	35.1
57.87	—4.08	7.7	37.7
57.45	—4.28	9.0	31.7
57.17	—2.03	8.9	32.5
56.47	—3.03	8.8	25.5
56.47	—3.33	7.2	25.2
55.88	—4.17	5.1	26.0
56.43	—3.50	5.2	25.3
55.88	—3.00	7.7	24.1
55.37	—4.55	4.5	23.5
55.28	—1.72	9.9	21.3
55.42	—2.77	7.0	21.2
54.83	—2.12	9.5	19.5
54.70	—2.43	9.0	18.7
54.53	—3.53	5.9	15.9
53.41	—3.00	7.7	13.6
53.18	—4.53	4.2	14.4
52.83	—4.30	3.0	11.5
52.40	—0.87	12.8	9.8
51.68	—1.03	13.6	7.1
52.08	—2.22	10.2	7.9
51.45	—2.57	10.1	4.7
51.68	0.37	17.2	7.3
51.10	1.22	19.9	5.7
50.78	0.30	16.5	3.4
50.63	—1.30	14.1	2.1
50.62	—3.50	7.5	1.3
51.37	—3.13	8.3	4.9
50.28	—4.13	5.5	0.5
50.05	—5.45	0.9	0.5
51.82	—4.83	3.1	4.3
52.60	1.08	18.3	12.0
52.92	0.05	14.6	12.6
53.38	—1.45	11.0	12.9
52.27	—2.68	8.2	8.5
53.62	0.03	14.4	15.5
53.83	—1.55	10.9	14.4

*In decimal degrees; ** in inches.

Table VI defines a projection after the fact. The problem is not to find a method of moving from the sphere (or earth) to a map, as is the usual case in the study of map projections, rather we have here a *fait accompli*. Although largely neglected in the literature this situation occurs frequently in practice. For example a satellite image is a representation of the surface of the earth on a flat surface. Frequently this transformation cannot be specified analytically in advance but can only be defined *a posteriori* by a table such as is given here for the Gough map. The objective in such a case may be to find the inverse table, i. e., given the plane coordinates of a place find its spherical coordinates. The theory is that if x = f (φ, λ) and y = g (φ, λ) then the inverse functions are related to these through the Jacobian determinant, assumed non-zero (18). Thus we again should know the partial derivatives. In practice a smooth inverse interpolation is used. An inverse of this type is also useful when solving problems using cartograms (19).

Frequently one does not know the projection of a map with which one must work. In such an instance a local projection may be invoked (20), or one must resort to a least squares fit to some assumed projection, or one uses a non-parametric interpolation procedure with a table such as Table VI. An exactly comparable situation exists in studies of the perception of the environment when people are required to draw, or otherwise specify, maps of their surroundings (21). The data can frequently be given in the form of a table associating true locations with estimated locations. These tables can then be treated as are conventional map projections, searching for properties preserved, for distortions, etc. (22). But since these transformations are not defined by explicit equations a numerical approach is more appropriate. Modern computing facilities render such an approach feasible.

[+]Pages 51-64 of E., Kretschmer, ed., Studies in Theoretical cartography, Deuticke, Vienna, 1977.

References

(1) Lambert, J.: Notes and Comments on the Construction of Terrestrial and Celestial Maps, 1772. Introduced and Translated by W. Tobler, Geography Department, University of Michigan, Ann Arbor, 1972, 125 pp.

(2) Airy, G.: "Explanation of a projection by balance of errors ...", Phil. Mag. 22, 409-421, 1861.

(3) Young, A.: Some Investigations in the Theory of Map Projections. London, Royal Geographical Society, 1920, 76 pp.

(4) Mescheryakov, G.: "The Problem of Choosing the most Advantageous Projections". Geod. Aerophotogr., 4, 263-268, 1965.

(5) Biernacki, F.: Theory of Representation of Surfaces for Surveyors and Cartographers. Translated from Polish, U.S. Department of Commerce, Washington D.C., 1963.

(6)Tobler, W.: Selected Computer Programs. Department of Geography, University of Michigan, Ann Arbor, 1970, pp. 39-48.

(7)Torgerson, W.: Theory and Methods of Scaling, New York, J. Wiley, 1938, pp. 255-258.

(8) Wolf, P.: "Horizontal Position Adjustment". Surveying and Mapping, XXIX, 4, 1969, pp. 635-644.

(9) Shepard, R.: "The Analysis of Proximities", Psychometrika, 27, 1962, pp. 125-140, 219-247.

(10)Kendall, D.: "Construction of Maps from Odd Bits of Information". Nature, 231, 1971, pp. 158-159.

(ii) Tobler, W.: "Mathematical Map Models". In the press.

(12)Akima, H.: "A Method of Bivariate Interpolation and Smooth Surface Fitting Based on Local Procedures". Communications, ACM, Vol. 17, No. 1, 1974, pp. 18-20, 26-31.

(13)Tobler, W.: "The Hyperelliptical and other new Pseudo-Cylindrical Map Projections". Journal of Geophysical Research, 78, 11, 1973, pp. 1753-1759.

(14)Tissot, M.: Mémoire sur la representation des surfaces et les projections des cartes géographiques Paris, 1881, 260 pp.

(15)Parsons, E.: The Map of Great Britain Circa A. D. 1360 known as the Gough Map, Oxford,

University Press, 1958, 38 pp.
(16)Tobler, W.: "Medieval Distortions: The Projection of Ancient Maps". Annals, Association of American Geographers, 56, 1966, pp. 351-360.
(17)— "Lattice Tuning". Ann Arbor, 1976, manuscript, 14 pp.
(18)Kaplan, W.: Advanced Calculus. Reading, Addison-Wesley, 1952, pp. 92-99.
(19)Tobler, W.: "A Continuous Transformation Useful for Districting". Annals, New York Acad. Sci., 219, 1973, pp. 215-220.
(20)— "Local Map Projections". The American Cartographer, 1, 1, 1974, pp. 51-62.
(21)Franckowiak, E.: "Location Perception and the heirarehical Structure of Retail Centers". Ph.D. dissertation, University of Michigan, Ann Arbor, 1973, 219 pages.
(22)Tobler, W.: "The Geometry of Mental Maps". In R. Golledge and G. Ruston, Spatial Choice and Spatial Behaviour. Columbus, Ohio State University, 1976.

Figure I: Errors as a function of the available constants.

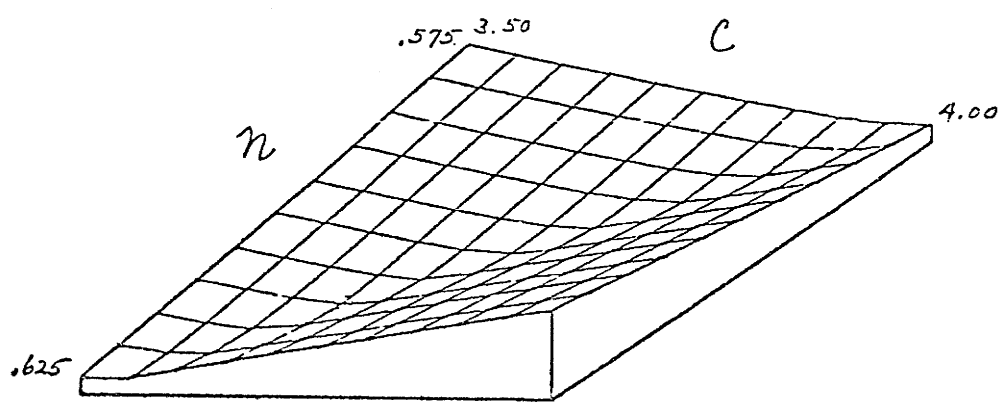

Figure II: An Empirical Map Projection of the United States, preserving spherical distances.

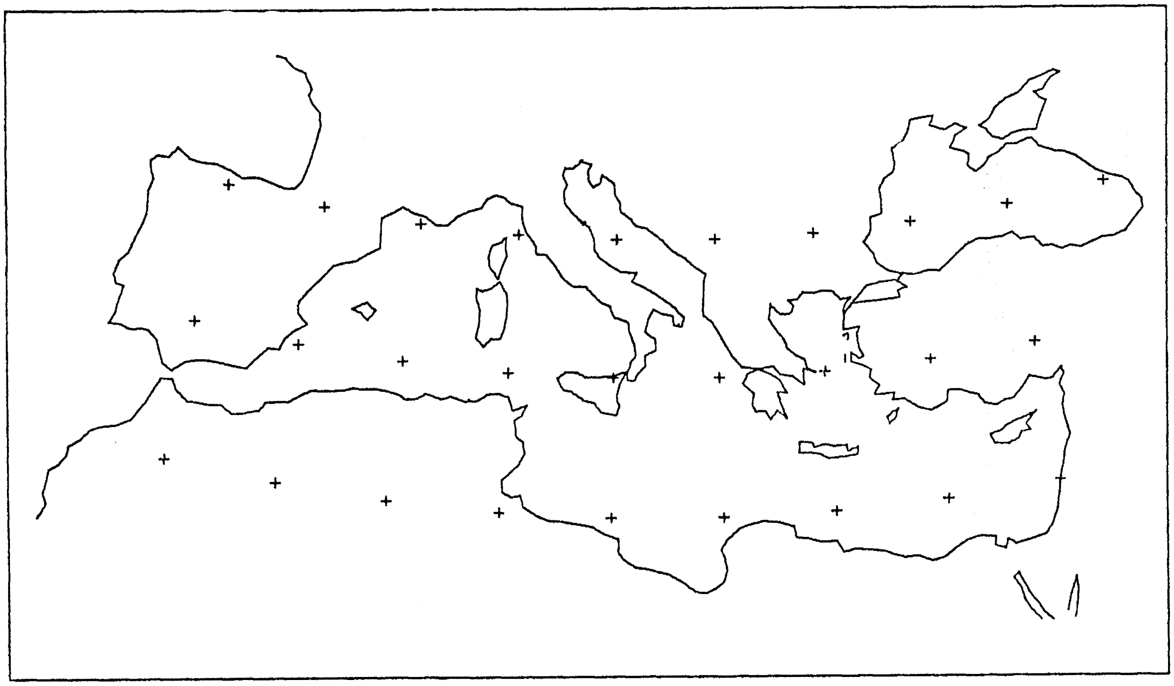

Figure III: Mediterranean map preserving loxodromic distances