

Thirty Five Years of Computer Cartograms

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The notion of a cartogram is reviewed. Then, based on a presentation from the 1960s, a direct and simple introduction is given to the design of a computer algorithm for the construction of contiguous value-by-area cartograms. As an example, a table of latitude/longitude to rectangular plane coordinates is included for a cartogram of the United States, along with Tissot's measures for this map projection. This is followed by a short review of the subsequent history of the subject and includes citation of algorithms proposed by others. In contrast to the usual geographic map, the most common use of cartograms is solely for the display and emphasis of a geographic distribution. A second use is in analysis, as a nomograph or problem-solving device similar in use to Mercator's projection, or in the transform-solve-invert paradigm. Recent innovations by computer scientists modify the objective and suggest variation similar to Airy's (1861) "balance of errors" idea for map projections. *Key Words:* *Anamorphoses, cartograms, map distortion, map projections, quasiconformal.*

The first reference to the term "cartogram" that I have found is to Minard, as follows: "In 1851 Minard published a series of maps called 'cartogrammes a foyer diagraphiques' or maps with diagrams" (Friis 1974, 133). The term is not listed in Robinson's (1967) paper on Minard, nor in his book on the history of thematic cartography (Robinson 1982). He only briefly mentions the related choropleth maps by name although he gives illustrations of several such maps from the mid-1800s. He also gives a map by Minard from 1850 in which deliberate distortion occurs in order to make room for the symbols on an illustration depicting flow (also see Robinson 1967, 101–02). For more detail, see Palsky (1996, 112–34). The first entry in Wallis and Robinson's (1987) survey of *Cartographical Innovations* is the term "cartogram," but they do not give the derivation. Wright uses the term in his introduction to Paullin's historical atlas of the United States (1932, xiv) and comments, "A cartogram is a cartographic outline upon which are drawn statistical symbols that do not conform closely to the actual distribution of the phenomena represented." He is, of course, referring to what is now called a statistical map. Kretschmer, Dörflinger, and Wawrik (1986, 1: 396), in their history of cartography, under the heading "Kartogramm" mean a choropleth or statistical map, but also refer to a "verzerrte" (distorted) map by Wiechel from 1903. There is also a reference to a map of Germany from 1903 in which statistics are shown on a schematic map (see Mayet 1905). In this country the *Washington Post* on Sunday, November 3, 1929, printed a map of the United States with state areas equal to population and taxation, accompanied by a proposal to the Congress to modify the

allocation of tariffs (Figure 1). This map would now be called a contiguous cartogram and shows Grundy's home state of Pennsylvania enlarged, as are the industrial states of Illinois, Michigan, New York, New Jersey, and Ohio.

The term "cartogram" was used repeatedly by Funkhouser in his history of graphical methods (1937). But by cartogram he means what we now call choropleth maps. Raisz (1938), in the first American cartography textbook, *General Cartography*, states that the term "cartogram is subject to many interpretations and definitions." He continues, "Some authors, especially in Europe, call every statistical map a cartogram, because it shows the pattern of distribution of a single element," in contradistinction to a topographic map, which combines many elements. He then has a section, as follows:

Value-Area Cartograms. In these cartograms a region, country, or continent is subdivided into small regions, each of which is represented by a rectangle. This rectangle is proportionate in area to the value which it represents in certain statistical distributions. The regions are grouped in approximately the same positions as they are on the map.

—(*General Cartography*, 257)

In this presentation, as in his earlier papers from 1934 and 1936, Raisz only refers to rectangular cartogrammatic diagrams. Raisz (1934, 292) also asserts that "the statistical cartogram is not a map." In his textbook *Principles of Cartography* (1962, 215) Raisz states that "A *cartogram* may be defined as a *diagrammatic* map." This suggests that the term "cartogram" is a contraction of the two words, given the flexibility of our languages in constructing such combinations.

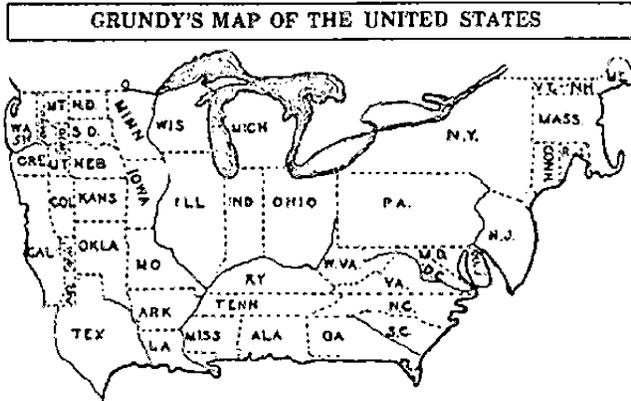


Figure 1. "Joseph R. Grundy, Pennsylvania manufacturer, suggested in the Senate lobby committee that the present equal power of States in voting on tariff bills is unfair because of differences in voting strength. Here's a map of the United States showing the size of each State on the basis of population and Federal Taxes." From the Washington Post November 3, 1929.

Interestingly, Funkhouser's paper actually presents a map that now might be called an area cartogram. This is a map of the countries of Europe in which each country is represented by a square whose size is proportional to the area of the country and with countries in their approximately correct position and adjacency (Figure 2). Could this be called an equal-area map? Or is it an equal-area cartogram? The map is dated 1870 and was used by the French educator Levasseur in one of his textbooks.

Raisz (1934, 1936) also has such "equal-land-area" rectangular cartograms of the United States, as well as

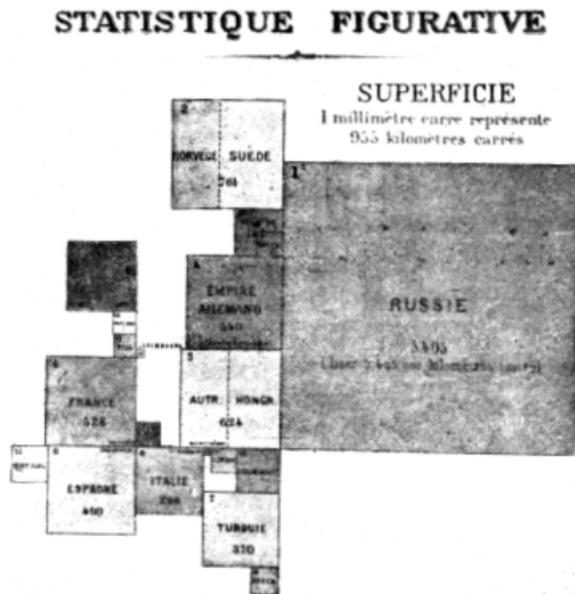


Figure 2. Levasseur's cartogram of Europe showing countries in their correct size after Funkhouser.

some displaying other phenomena (Figure 3). Looking at these examples it is clear that one should distinguish between cartograms that rigorously maintain the correct adjacency table (O'Sullivan and Unwin 2002, 40, 155) and those that do not. Diagrams such as those of Levasseur and Raisz have adjacency tables that have both too many and too few adjacencies. In those that do display the correct topological adjacency, it is worth noting whether or not the maps are continuous, with continuous partial derivatives, or only piecewise continuous.

A discussion of "value-by-area" cartograms can now be found in several contemporary cartographic textbooks, for example, Dent (1999, 207–19) and Slocum (1999, 181–84). In Canters's recent book on map projections (Canters 2002, 157–67), they are included as "variable scale maps." The French use the term "anamorphose" (for a derivation

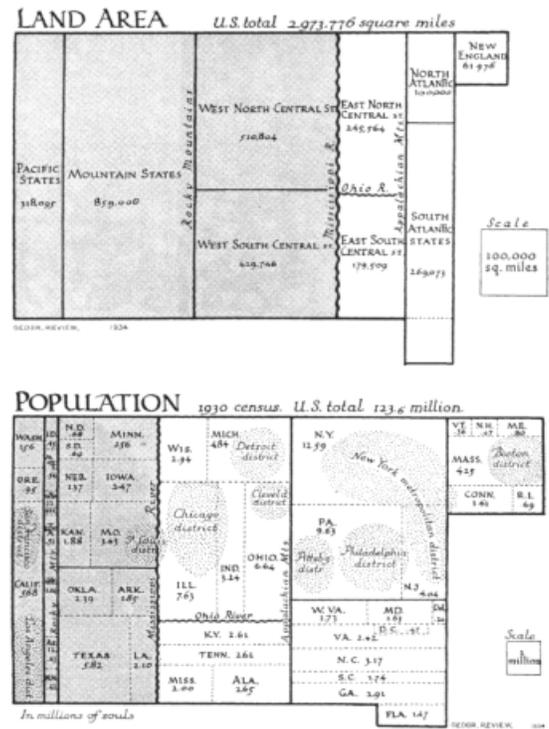


Figure 3. Two of Raisz's (1934) rectangular cartograms. The top diagram is a statistical cartogram showing the land surface (km²) by Census divisions—an equal area map. Below this is a "Rectangular statistical cartogram with rectangles representing geographical divisions of the Census and states proportionate in size to their population." Comparing the differential position of the Appalachian Mountains and the Mississippi River on the two diagrams dramatically illustrates the differences. Additional cartograms in Raisz's paper include representations of "National Wealth," "Value Added by Manufacture," "Farm Products," "Mine & Quarry Products," "Crude Oil," "Sand & gravel," and "Natural Gas," all scaled in 1920 dollars. From Raisz (1934), "The Rectangular Statistical Cartogram," *Geographical Review*, 24(2): 293. Used by permission of the American Geographical Society.

of this term see Hankins 1999), in German-speaking countries “verzernte Karte” is most prevalent, and the Soviets have used the word “varivalent” maps. These terms are a bit misleading since the scale on a geographic map is never a single constant, though the meaning here is clear.

I became interested in the subject in 1959 and my doctoral thesis on *Map Transformations of Geographic Space* (Tobler 1961) was devoted to this topic. There are now two generally accepted types of cartograms. One form stretches space according to some metric different from kilometers or miles, such as cost or time. These are not considered in this review, although they constituted the bulk of the above-cited thesis. I also do not consider the piecewise continuous (noncontiguous) cartograms described by Olson (1976). The concern here is with the type that stretches space continuously according to some distribution on a portion of the earth’s surface. These are generally referred to as area (or areal) cartograms, or (following Raisz) as value-by-area maps. The single chapter of my dissertation that was devoted to this topic was published in the *Geographical Review* (Tobler 1963), but with a mathematical appendix that was not in the thesis. When I arrived at the University of Michigan in Ann Arbor, I began to develop computer programs that could compute area cartograms. The easiest way for me to introduce this subject is to reproduce a lecture given in the 1960s to Howard Fisher’s computer graphics group at Harvard University. The following is a transcription of the original overhead viewgraphs from that lecture. It remains a simple and clear definition of the problem and of one solution.

The Harvard Presentation

A value-by-area cartogram is a map projection that converts a measure of a nonnegative distribution on the earth to an area on a map. Consider first a distribution $h(u, v)$ on a plane (Figure 4):

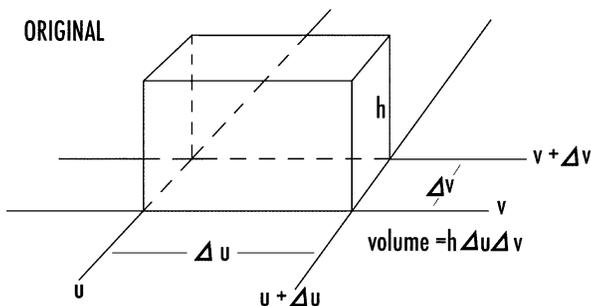


Figure 4. A representation of the to-be-preserved density shown in a perspective diagram.

Next consider an area on the map (Figure 5):

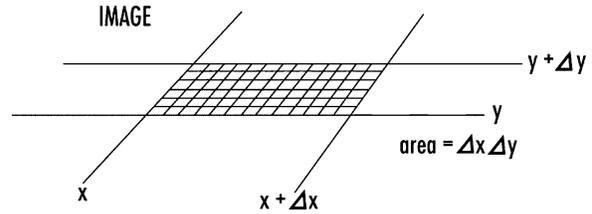


Figure 5. The density converted to a plane area.

We want these to be the same. That is, the map image is to equal the original measure: *image area on map equals original volume on surface*, or $\Delta x \Delta y = h \Delta u \Delta v$.

As an aside, observe that in his treatise of 1772 J. H. Lambert defined an equal-area map projection in exactly this fashion, setting spherical surface area equal to map surface area, of course with the cosine of the latitude included to account for a spherical earth.

Now replace the Δ 's by d 's, that is, $dx dy = h du dv$. This is for one unit area but it can be rewritten in integral form to cover the entire domain as

$$\int \int dx dy = \int \int h du dv.$$

To solve this system we can insert a transformation, that is, divide both sides by $du dv$ to get $dx dy / du dv = h(u, v)$. This can be recognized as the introduction of the Jacobian determinant, as covered in beginning calculus courses. With this substitution the condition equation becomes

$$J = \partial(x, y) / \partial(u, v) = h(u, v).$$

Written out in full we have the equation

$$\partial x / \partial u \partial y / \partial v - \partial x / \partial v \partial y / \partial u = h(u, v). \quad (1)$$

To apply this mildly nonlinear partial differential equation to a sphere it is only necessary to multiply by $R^2 \cos \phi$ on the right hand side of this equation, substituting longitude (λ) and latitude (ϕ) for the rectangular coordinates u and v . That a valid value-by-area preserving map can exist follows from the solution properties of this differential equation.

Reverting now to the pictures on the plane, a small rectangle will have nodes identified by Cartesian coordinates given in a counterclockwise order (Figure 6). The area, A , of such a rectangle is given by the determinant formula

$$2A = \begin{vmatrix} X_1 & Y_1 \\ X_2 & Y_2 \end{vmatrix} + \begin{vmatrix} X_2 & Y_2 \\ X_3 & Y_3 \end{vmatrix} + \begin{vmatrix} X_3 & Y_3 \\ X_4 & Y_4 \end{vmatrix} + \begin{vmatrix} X_4 & Y_4 \\ X_1 & Y_1 \end{vmatrix}$$

It is now desired that this area be made equal to the “volume” $h(u, v)$. This can be done by adding increments

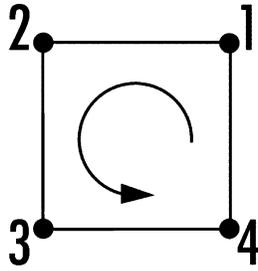


Figure 6. A square with vertex identification. Counterclockwise numbering is indicated.

Δx , Δy to each of the coordinates on the map. The new area can be computed from a determinant formula as above but now using the displaced locations $X_i + \Delta X_i$, $Y_i + \Delta Y_i$, $i = 1, \dots, 4$ (Figure 7).

Call the new area A' ("a prime"). Now use the condition equation to set the two areas equal to each other, $A' = A$. Recall that A' is, by design, numerically equal to $h(u, v)$. The equation $A' = h = A$ involves eight unknowns, the ΔX_i and the ΔY_i , $i = 1 \dots 4$ for the rectangle (Figure 8). Now it makes sense to invoke an isotropicity condition to attempt to retain shapes as nearly as possible. So set all ΔX_i and ΔY_i equal to each other in magnitude, and simply call the resulting value Δ . That is, assume

$$\begin{aligned} \Delta X_2 &= -\Delta X_1 & \Delta Y_2 &= -\Delta Y_1 \\ \Delta X_3 &= -\Delta X_1 & \Delta Y_3 &= -\Delta Y_1 \\ \Delta X_4 &= -\Delta X_1 & \Delta Y_4 &= -\Delta Y_1 \end{aligned}$$

and finally that $\Delta = \Delta X_1 = \Delta Y_1$ (Figure 9). This is also the condition that the transformation be, as nearly as possible, conformal, that is, locally shape preserving, and minimizes the Dirichlet integral

$$\int_R (\partial x^2 / \partial u + \partial y^2 / \partial v + \partial x^2 / \partial v + \partial y^2 / \partial u) \, du \, dv.$$

And this renders the transformation unique. Thus A' is just an enlarged (or shrunken) version of A , a similitude. Working out the details yields

$$A' = 4\Delta^2 + \Delta(X_1 - X_2 - X_3 + X_4 + Y_1 + Y_2 - Y_3 - Y_4) + A.$$

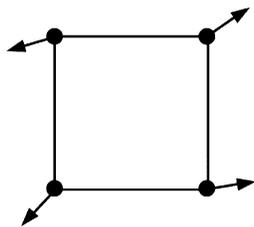


Figure 7. Possible vertex displacements, with no constraints.

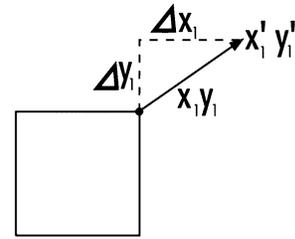


Figure 8. Notation for the displacement of one vertex.

This quadratic equation is easily solved for the unknown Δ . Once this quantity is found, the problem has been solved, but for only one piece of territory. The map area A' is now numerically equal to the numerical value of $h(u, v)$ for that piece.

Many different equal-area map projections are possible, and the same holds for value-by-area cartograms. There is one defining equation—(1) above—to be satisfied, but this is not sufficient to completely specify a map. Two conditions are generally required to determine a map projection, or the differential equation must be given boundary conditions. The choice of a transformed map that looks as nearly as possible like the original was made above, and this minimizes the angular distortion, locally preserving shape as nearly as possible. Sen (1976) presents another possible condition. In his example, based on population for the United States, he retains the latitude lines as equally spaced horizontal parallel lines (a type of pseudocylindrical projection?), but the example, while preserving the desired property, is hardly legible. It is also possible to formulate the problem in polar coordinates, as for azimuthal map projections. Three different versions of this possibility are given in Tobler (1961 and 1963, 1973 and 1974, and 1986).

If the regions with which one is dealing are irregular polygons, instead of rectangles, the procedure is exactly the same. One simply translates to the centroid of a polygon and then expands (or shrinks) by the proper amount to get the desired area size, getting a local

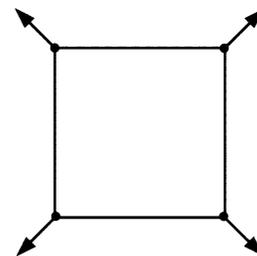


Figure 9. Uniform displacement of all vertices, yielding a similarity transformation.

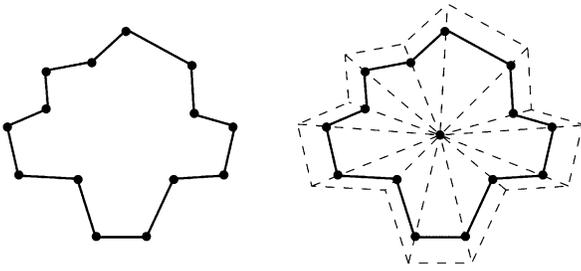


Figure 10. A polygon and displacements to yield a similarity transformation by expansion from the centroid.

similitude. In order to do this, use the same reasoning as above, expanded to cover general polygons instead of rectangles. The result is that the scale change is the square root of (A'/A) , from which Δ is easily calculated (Figure 10).

When two rectangles, or two polygons, are attached to each other, they will have nodes in common. Then the amounts of displacement calculated independently for a single node associated with more than one area will differ. Suppose one displacement is calculated to be Δ_1 and the other Δ_2 . Then take the (vector) average. But this means that neither will result in the desired displacement. In particular, in a set of connected rectangles, or polygons, this problem will occur at almost all nodes. And some nodes will be connected to more than two regions (Figure 11). Again, just average all of the displacements. After all of these displacements have been calculated apply them by adding the increments to every node. Then repeat the process in a convergent iteration. Eventually, all regions will converge to their proper, desired size.

There remains one final problem. The displacement at a node can be such that it requires that the node cross over the boundary of some region. This must be prevented.

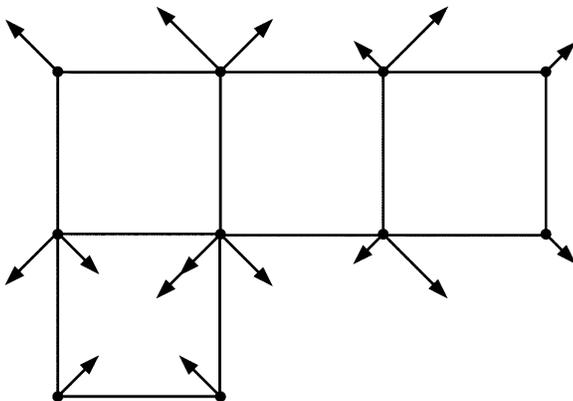


Figure 11. Illustrating conflicting displacements. One node may need to be displaced to two (or more) different positions. The resolution of this conflict is described in the text.

Equally seriously, displacing two nodes may result in the link between them crossing over some other node. And this again must be prevented. Both of these situations are illustrated in the accompanying figure (Figure 12).

Underrelaxation—shrinking the displacements to some fraction, say 75 percent, of the desired values—helps avoid, but does not prevent, the problem. The technique used in my computer programs to solve this difficulty was to look at all adjacent nodes, three at a time. This triple makes up a triangle. When a node of the resulting triangle is displaced to cross over the opposite edge of the triangle, as in Figure 13, then the problem (a node crossing over a line, or a line crossing over a node) has occurred. Turning a triangle inside out in this manner changes the algebraic sign of the triangular area, from positive to negative, and this makes detection of the difficulty very simple. When this happens, the displacement must be adjusted. Therefore, if a crossing is detected in the triangle, then the displacement is shrunken until it is no longer is a problem. Thus every node must be checked against every boundary link, and every link must be compared to every node. This can get quite tedious, consuming considerable computer time. This topological checking slows the algorithm down considerably. The next iteration computes a new set of displacements and the desired result is eventually achieved, in a convergent iteration. The topological check also prevents negative areas from occurring. Negative areas are not permitted, since by assumption, $h(u, v)$ is nonnegative.

The “error,” the total discrepancy between the desired result and the result obtained, is measured by $\sum |A' - A| / \sum A$ with A' normalized so that $\sum A' = \sum A$ over all areas. The convergence of the algorithm then follows a typical monotonic exponential decay. My experience on an IBM 709 computer, with data given by latitude and longitude quadrilaterals and using one degree population data (a 25 by 58 lattice of 1450 cells) for the continental United States, was that the program required 25 seconds per iteration and required about 20 iterations. Today’s computers would require about one second per iteration with this algorithm. Using the 48 contiguous U. S. states as data-containing polygons takes considerably less time since there are fewer areas (only 48 cells) to be evaluated.

The Computer Programs

The exact date of the Harvard presentation is not available, but Howard Fisher retired from his position as lab director in 1967 so it must have been before that. Three computer versions then existed; one program treated data given by latitude/longitude quadrangles,

Figure 12. Illustrating a vertex crossing a line, and a line crossing a vertex. Both are violations of the topology.



another was for irregular polygons, and the third assumed that the geographical data are represented by a mathematical equation. The lattice version worked on the latitude/longitude grid (or any orthogonal lattice) and produced a table of x, y coordinates for each point of latitude and longitude. In other words the program produced tables of $x = f(\varphi, \lambda)$, $y = g(\varphi, \lambda)$. These are the two equations required to generate and define a map projection. An example is given in the table (Table 1). Using a standard map projection program, it is possible to plot coastlines, state boundaries, rivers, and so forth, by interpolation using these coordinates. Today most map projections are obtained directly from known equations, Robinson's projection being a notable exception, requiring both a table lookup and an interpolation. Most map projections are not obtained by iteration, but Mollweide's projection is done in this manner to produce the table from which the projection is then calculated and interpolated. An advantage of presenting the cartogram as a table of geographic-to-rectangular coordinates is that this gives the final results of the iteration and that this table can be passed on for the use of others or to plot additional detail. All that is required for others to reproduce the cartogram is a simple interpolation routine. The Figures 14 and 15 show an example produced from the foregoing table at the University of Michigan using a plotter.

The cartogram program was also tested by entering the spherical surface area for latitude/longitude quadrangles and, as expected, yielded an equal-area map projection for the United States similar to that by Albers. As far as I am

aware no one else has tested their algorithm in this simple manner. All equal-area maps are a special case of cartograms in which the surface area is the property to be preserved, as is easily seen from the equations.

A separate program, again independent of any particular map projection, computed the linear and angular distortion of Tissot's (1881) indicatrix by finite differences from this same coordinate file (Table 2). Tissot's measure of area distortion (his S) on map projections had been shown (Tobler 1961) to be equal to the desired area on a cartogram. Just as an infinite number of equal-area map projections are possible, there are many possible value-by-area cartograms. The choice depends on the additional conditions invoked. My choice was to come as close as possible to the conventional map by minimizing angular distortion. The Tissot evaluation program provides an objective measure as to how well this target is achieved.

The second cartogram program used state outlines, or general polygons, instead of a grid, directly. Both programs allowed one to specify particular points to be fixed—not to be moved—so that, for example, some exterior boundaries could be held constant or points critical for recognition of landmarks could be retained. Both programs also had an option to produce an initial pseudocartogram (Tobler 1986a), treating the quantity to be preserved, $h(u, v)$ as if it could be approximated by a separable function of the form $h_1(u) h_2(v)$. To do this, the program “integrated” the data in the u direction and then separately in the v direction. The effect is somewhat like using a rolling pin in two orthogonal directions to flatten a batch of bread dough. This could be used to compute a beginning configuration from which to initiate the iterations, saving computer time, but it also would affect the resulting appearance of the cartogram. A similar residual effect can be observed if a cartogram is begun with information given on any map projection of the area of interest. For example, the results are affected if plane coordinates from the Plate Carée (rectangular) projection, the sinusoidal projection, or that of Carl Mollweide is used as the initial configuration from which to compute a cartogram of the world, or any part thereof. Using Tissot's results, integrated over the entire map, one can distinguish between, and rank, these alternative cartogram configurations.

Another trick is to begin with a simplified set of polygon outlines, iterate to convergence, and then restart with

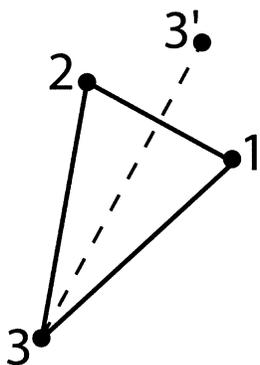


Figure 13. Adjusting the topology by checking the algebraic sign of the area of a triangle.

Table 1. Map Projection Coordinates for a Cartogram of the United States

LATITUDE		24 N	29 N	34 N	39 N	44 N	49 N
LONGITUDE							
125 W	X	0.901	0.512	0.258	0.916	1.469	0.705
	Y	0.520	1.295	6.529	17.856	23.559	28.786
120 W	X	2.572	1.744	1.428	3.050	3.350	3.772
	Y	0.272	1.151	7.256	16.240	22.415	29.166
115 W	X	5.364	5.634	5.422	4.351	4.779	5.505
	Y	0.384	1.203	7.656	14.756	22.841	27.355
110 W	X	6.457	6.526	6.196	5.171	6.072	6.231
	Y	0.519	1.643	9.624	15.592	23.570	27.464
105 W	X	7.124	7.118	6.897	6.082	6.783	7.151
	Y	0.526	1.908	10.340	16.850	23.989	27.529
100 W	X	8.375	8.474	8.035	7.875	8.102	7.985
	Y	0.177	1.526	9.030	17.452	23.991	27.597
95 W	X	12.221	13.372	10.826	10.653	9.944	10.867
	Y	0.205	1.873	9.683	17.012	24.102	28.341
90 W	X	16.575	16.741	17.082	14.738	15.994	17.012
	Y	0.481	1.492	8.323	16.218	25.606	28.773
85 W	X	23.867	24.124	23.866	24.013	24.001	24.279
	Y	0.325	1.401	6.476	14.864	26.785	28.610
80 W	X	33.333	33.993	32.520	32.508	33.131	32.813
	Y	0.618	2.031	4.538	13.440	26.556	28.511
75 W	X	41.886	42.043	43.210	40.877	39.962	39.961
	Y	2.195	2.971	4.184	8.479	26.184	28.638
70 W	X	46.932	46.854	47.618	48.131	48.249	47.535
	Y	2.085	2.958	5.590	10.501	26.452	29.172
65 W	X	48.652	48.117	48.811	49.469	49.630	49.291
	Y	1.311	2.718	6.350	13.588	25.608	28.695

Values in degrees

more detailed polygons. This process can be automated, and also works for data in gridded form, going from low to high resolution. Furthermore, since substantive information given by political units (polygons) often varies dramatically from one polygon to the next, it also makes sense to use pycnophylactic reallocation, which does not

change the total data within any polygon but redistributes it to obtain a smoother arrangement. Thus there is less drastic fluctuation from polygon to polygon. If using finite differences (Tobler 1979b) for this smooth reallocation, the individual polygons are partitioned into small quadrangular cells, and the computer version for a regular

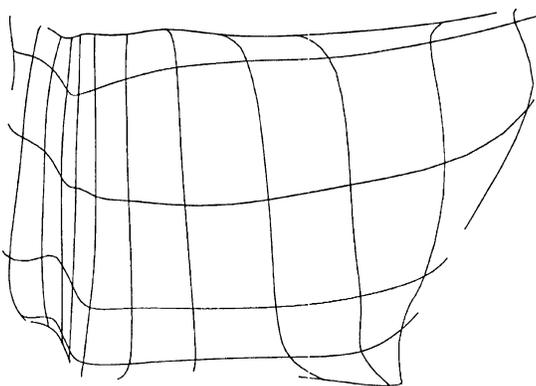


Figure 14. The latitude-longitude graticule for a cartogram of the United States. Computed by the author.

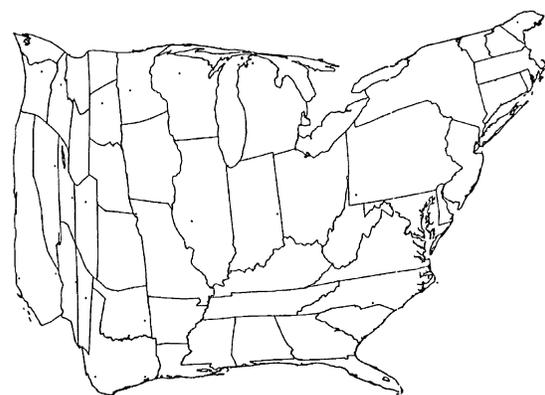


Figure 15. The United States drawn according to the projection graticule of Figure 14. Computed by the author.

Table 2. Tissot's Measures of Map Projection Distortion for a Cartogram of the United States

Lat	Lon	a	b	k	h	2 ω
29	-120	0.99697	0.38274	0.38725	0.99522	52.8709
29	-115	1.27582	0.29539	0.33808	1.26517	77.2165
29	-110	1.20178	0.21477	0.26406	1.19192	88.3369
29	-105	1.06845	0.23102	0.23704	1.06713	80.2467
29	-100	1.01472	0.36120	0.36165	1.01456	56.7153
29	-95	1.17550	0.86628	0.92103	1.13310	17.4216
29	-90	1.75168	1.11213	1.63623	1.27593	25.8085
29	-85	1.96759	1.36739	1.96237	1.37486	20.7360
29	-80	1.82679	1.50669	1.82620	1.50741	11.0206
29	-75	2.02536	1.71991	1.72856	2.01798	9.3559
29	-70	1.87223	1.10027	1.10735	1.86805	30.1049
34	-120	1.56400	0.44755	0.55812	1.52804	67.4246
34	-115	1.52253	0.25186	0.26751	1.51986	91.4696
34	-110	1.41802	0.31432	0.40560	1.39466	79.1535
34	-105	1.38388	0.32354	0.39587	1.36495	76.7817
34	-100	1.49628	0.44266	0.44323	1.49611	65.8309
34	-95	1.46455	0.80596	0.84110	1.44466	33.7238
34	-90	1.86459	1.47860	1.63220	1.73172	13.2598
34	-85	2.26480	1.92408	2.16948	2.03094	9.3309
34	-80	2.52269	1.79407	2.17508	2.20265	19.4347
34	-75	2.41680	1.66779	1.91753	2.22385	21.1328
34	-70	2.21083	0.96037	1.20576	2.08715	46.4467
39	-120	1.53085	0.64259	0.68012	1.51454	48.2452
39	-115	0.98584	0.62055	0.68503	0.94218	26.2880
39	-110	1.43964	0.21640	0.39408	1.40147	95.2331
39	-105	1.49638	0.23560	0.24867	1.49426	93.4289
39	-100	0.95349	0.49528	0.51386	0.94360	36.8755
39	-95	1.59407	1.09007	1.16768	1.53812	21.6453
39	-90	1.90122	1.25518	1.72668	1.48616	23.6209
39	-85	2.08889	1.27772	2.04533	1.34635	27.8847
39	-80	2.52611	1.10685	2.50652	1.15052	45.9913
39	-75	1.94916	0.55709	1.94746	0.56301	67.4824
39	-70	0.94526	0.53746	0.77273	0.76503	31.9279
44	-120	1.01444	0.55491	0.91442	0.70771	34.0527
44	-115	0.74517	0.64821	0.66829	0.72722	7.9806
44	-110	1.04047	0.16828	0.53028	0.91087	92.3680
44	-105	1.33361	0.22041	0.92894	0.98191	91.5060
44	-100	0.89847	0.85648	0.86942	0.88595	2.7421
44	-95	1.65079	0.58353	1.46564	0.95787	57.0667
44	-90	1.49627	0.78560	1.49614	0.78584	36.2918
44	-85	2.39956	0.61478	2.39948	0.61510	72.6117
44	-80	2.50103	0.39753	2.50058	0.40034	93.0551
44	-75	1.79856	0.18823	1.79253	0.23894	108.2926
44	-70	0.84720	0.35218	0.84512	0.35715	48.7523

Tissot's "a" measures the maximum linear stretch, "b" is the minimum linear stretch, "h" and "k" are the linear stretch along the meridians and parallels, "2 ω " is the maximum angular distortion. The areal distortion (Tissot's S) is the product of "a" times "b" and is, by design, proportional to the population. Other properties can be computed from these basic entities.

lattice is used. The plane coordinates of the cell vertices are then retained for subsequent plotting. If finite elements with triangles (Rase 2001b) are used to implement the smoothing reallocation within the polygons, then a computer program version for irregular areas, perhaps specialized for triangles, can be used.

In addition to the Harvard lecture, the same algorithm was also presented to a conference on political districting (Tobler 1972) and at a conference on computer cartography applied to medical problems (Tobler 1979a). A description of how to proceed when the geographical arrangement of phenomena is described by an approx-

imating mathematical equation was also given in the 1961 thesis, in the 1963 paper, in the 1974 program documentation, and in the 1979 paper. An equation in two variables can depict a geographical distribution with a high degree of accuracy, or, in a simpler version, can be used to just describe an overall trend; either representation is appropriate for the construction of a cartogram. In the case of an approximating trend, the exact value-by-area property may not be obtained since only the trend is represented by the map and the cartogram does not retain correct area values as desired but only approximates them. The approximation depends on the fit of the trend to the data. When such a descriptive equation is available, it may no longer be necessary to use an iterative procedure; the cartogram might be computed directly, depending on the complexity of the equation. For simple examples and a related idea, see Monmonier (1977).

Additional programs at that time could produce a hexagonal grid to cover a region and to produce an inverse transformation from the latitude/longitude grid and the tables of x, y coordinates. The resulting tables are then for $\varphi = f^{-1}(x, y)$ and $\lambda = g^{-1}(x, y)$. This allows the inverse transformation of the hexagonal grid to lie, in warped form, over the original image, in an attempt to replicate Christaller's central place theory in a domain of variable population (Figure 16). Only one of the competing computer algorithms (see below) is known to allow complete inversion of a cartogram.

The several FORTRAN programs for cartograms mentioned above were distributed by me in a 110-page Cartographic Laboratory Report (No. 3, 1974) from the

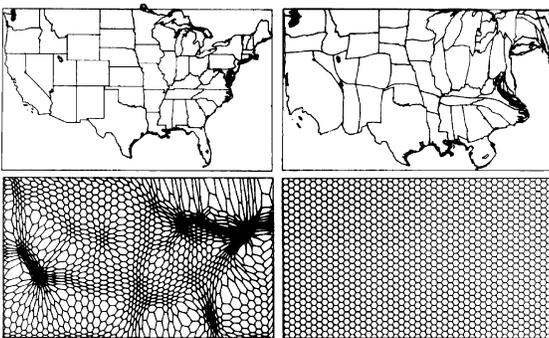


Figure 16. Illustrating the transform-solve-invert paradigm in an analytical use of a cartogram. The bottom two diagrams are intended to be superimposed on the top two. At the top right is a population cartogram of the United States. Below this is a uniform set of hexagons. Using the inverse transformation the hexagons are converted to cover the United States approximately partitioning this area into zones of equal population. Tissot's results are valid in both directions, consequently the inverse hexagons approximate his indicatrix. After Tobler (1973). Used by permission of the New York Academy of Sciences.

geography department at the University of Michigan. Several of these programs were later (circa 1978) implemented on a Tektronix 4054 in the geography department at the Santa Barbara campus of the University of California.

An interactive version of the polygon program version, using a Tektronix display terminal, was prepared in 1970 by Stephen Guptill and myself, and presented at a computer conference (Tobler 1984). This program allowed the topological checking to be performed visually, thus avoiding the need for the tedious, triangular inside-outside computation by computer. The discrepancy in each polygon was displayed on an interactive computer screen and was indicated by scaled plus or minus symbols at the polygon centroids. The proposed change at each iteration could also be superimposed on the previous result or on the original configuration, shown as a set of dashed lines. Using an interactive cursor, the cartographer could zoom in on a location on the map to move and improve offending, or inelegant, node displacements. No numerical calculations were required by the cartographer. The computer costs (and time) were then reduced 100-fold since the topological checking could be disengaged. This was followed by further iterations, and interaction, to improve the fit to the desired areal distribution. An example is shown in the figure (Figure 17) for a portion of a map of South America. Williams (1978) and Torguson (1990) also developed interactive cartogram programs.

From Ruston to the Internet

In the following paragraphs several additional computer cartogram programs are briefly described or men-

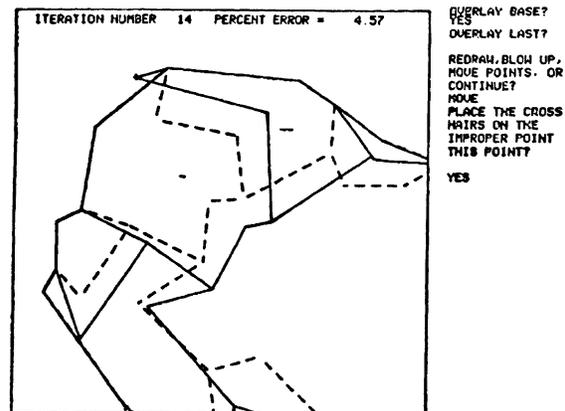


Figure 17. One frame from an interactive cartogram program. The different lines represent stages of the iteration. The cursor placement (plus sign) at a node indicates a location for potential change by the cartographer. Positive and negative symbols indicate areas that are too large or too small relative to the desired objective. Computed by the author.

tioned. The details for each program can be found in the citations. The intent here is only to give a short review of this literature.

In 1971 G. Ruston published a computer program that was based on a physical analogy. One may imagine that a thin sheet of rubber is covered with an uneven distribution of inked dots representing a distribution of interest. The objective is to stretch the rubber as much as necessary until the dots are evenly distributed on the sheet. This simple description is an approximate representation of the mathematical statement used for his computer program. If the dots represent the distribution of, say, population, the resulting cartogram is such that map areas are proportional to the population. When the rubber sheet is relaxed to its original prestretched form, hexagons previously drawn on the stretched surface can represent market areas. The implication is again to use the cartogram as a test of the Christaller theory of the distribution of cities in a landscape. Uniqueness would seem to me to depend on boundary conditions since the cartogram process involves solving a partial differential equation.

In 1975 A. Sen published a theorem about cartograms. In effect, he asserted that the least distorted cartogram has the minimal total external boundary length of all possible areal cartograms. Interestingly, this had been empirically discovered by Skoda and Robertson in 1972 while constructing a physical cartogram of Canada, using small ball bearings to represent the unit quantity. Similar manual methods were reported by Hunter and Young (1968) and by Eastman, Nelson, and Shields (1981).

Kadman and Shlomi, in 1978, introduced the idea that a map could be expanded locally to give emphasis to a particular area. This was based on rather ad hoc notions of the importance of an area and not on the matching to a particular distribution. Lichtner (1983) introduced similar concepts as did Monmonier (1977). Snyder's (1987) "Magnifying Glass" map projection uses a similar idea. Hägerstrand (1957) already did this, of course, but he based his map distortion on an anticipated distribution and distance decay of the location of migrants from a city in Sweden. This resulted in an azimuthal projection with enlargement at the center of interest. More recent versions of the concept appear in Rase (1997, 2001a), Sarkar and Brown (1994), and in Yang, Snyder, and Tobler (2000). An Internet firm (<http://www.Idelix.com>) now supports an interactive version of this technique with variable moving morphing windows.

In 1983 Appel, Evangelisti, and Stein of IBM patented a cartogram program that worked somewhat like a cellular automaton. Areas were represented as cells of a lattice and "grew" by changing state (color), depending on the need for enlargement or contraction.

A decade after my several computer programs were distributed, Dougenik, Nicholas, Chrisman, and Niemeyer of Howard Fisher's Harvard Computer Graphics Laboratory published (1985) an algorithm that differed from the one that I had developed in only a small, but important, respect. In my algorithm the displacements to all nodes in one iteration were applied, simultaneously, only after they had all been computed, and adjusted, for all polygons. That was at the end of one complete pass through the program. Another iteration of the same procedure then followed until the stopping criterion was satisfied. Instead of this, the Harvard group, after computing "forces" for only one polygon, applied them immediately to all nodes of *all* of the polygons. But these displacements, discounted by a spatially decreasing function away from the centroid of the polygon in question, were applied to all of the nodes of all polygons simultaneously. Thus the objective was approximately satisfied for only one polygon, but all polygons were affected by the "forces" and modified. They then moved on to the next polygon and repeated the procedure. Iterations were still required, with a stopping rule. The result is a continuous transformation of a continuous transformation, and that, of course, is continuous. As a consequence, they avoided most of the necessity for a tedious topological constraint and virtually all of the topological problems were avoided. Depending on the complexity of the polygon shapes, occasional overlapping might still occur, but only infrequently. Thus there is an improvement in speed, but not necessarily in accuracy. Almost all subsequently developed computer programs stem from this 1985 publication, which included pseudo-code.

In 1988 Tikunov of the Soviet Union presented a brief review of the history of cartograms and described several manual, mechanical, and electrical methods of production for these types of maps, along with a sketch of a computer method. This paper also included many references to the considerable Russian literature. A new mathematical algorithm, using Stoke's Theorem and line integrals, was also presented in 1993 by the Soviet authors Gusein-Zade and Tikunov. Another version, with emphasis on medical statistics, formed the basis for two Ph.D. dissertations at the University of California at Berkeley (Selvin et al. 1988, also described in Merrill, Selvin, and Mohr 1991, and Merrill 1998, with an extensive bibliography). In these medically oriented papers the important emphasis is on the analytical use of a cartogram as a new geometric space in which to do statistical testing, in spite of the fact that distances are not invariant under this transformation. The objective, hence, is not visual display but analysis. An earlier health-related paper, using a manual procedure and with a similar objective is by Levison & Haddon (1965). Merrill (1998) provides other early examples.

D. Dorling, in 1991, developed a novel approach. Using only centroids of areas he converted each polygonal area into a small bubble—a two dimensional circle. These bubbles are then allowed to expand, or contract, to attain the appropriate areal extent. At the same time they attempt to remain in contact with their actual neighbors. Dorling also colored the resulting circles, depending on some additional attribute. His later (Dorling 1996) work contains a rather detailed history of the subject and gives numerous examples of alternative types and, as of the 1996 date, includes the most comprehensive bibliography on the subject of area cartograms, along with two programs to compute cartograms. One of these programs was for polygons converted to a raster, and included an inverse procedure. The other program was for his bubble algorithm. This is most popular in the United Kingdom, perhaps because the complete computer program was published there, and now also seems to be available on the Internet. Recall that the U.K. statistical agencies at that time provided only centroids, and no boundary data, for small areas. He also (1995) published an atlas of social conditions making extensive and effective use of the display capabilities of his procedure. Dorling did not present any equal-area maps using this algorithm, with spherical land area (km^2) as the property to be preserved, except as initial configurations. Adrian Herzog of the Geography Department of the University of Zürich has also prepared a program for interactive use on the Internet (<http://www.statistik.zh.ch/map/mapresso.htm>). Two further implementations for use in conjunction with a GIS from ESRI have also been reported (Jackel 1997; Du 1999). An innovative undergraduate thesis has also been presented by Brandon (1978) done under the supervision of Eric Teicholz at Harvard, and another from the United Kingdom by Inglis (2001), and there is a master's thesis by Torguson (1990). Some literature exists in German (Elsasser 1970; Kretschmar 2000; Rase 2001a), and from France, too (Cauvin and Schneider 1989). A recent search on Google located 2,170 entries under "cartogram" on the Internet, including analytical applications in medicine (mostly epidemiology), uses for display in geography, and, most interestingly, a very large number of exercises for grade-school children. Also included on the Internet are animations, made possible by the iterative nature of the algorithms (an example, <http://www.bbr.uni.edu/cartograms>).

Recognition Difficulties

It has been suggested that cartograms are difficult to use, although Griffin (1980, 1983) does not find this to be the case. Nevertheless Fotheringham, Brunson, and

Charlton (2000, 26), in considering Dorling's maps, state that cartograms

"can be hard to interpret without additional information to help the user locate towns and cities."

The difficulty here is that many people approach cartograms as just a clever, unusual display graphic rather than as a map projection to be used as an analogue method of solving a problem, similar in purpose to Mercator's projection. Mercator's map is not designed for visualization and should not be used as such.

If the anamorphic cartograms are approached as map projections, then it is easy to insert additional map detail. In the case of Dougenik and colleague's, Dorling's, or other versions, simply knowing the latitudes and longitudes of the nodes or centroids allows one to draw in the geographic graticule or to display any additional data given by geographic coordinates. And this can be done using a standard map projection program augmented by a subroutine to calculate a map projection given as a table of coordinates. Included here, in addition to the geographic graticule, could be roads, rivers, lakes, and so on, which would enhance recognition. Dorling's bubbles could be replaced by boundaries of larger administrative units, or other features. Thus the "bubbly" effect need not be retained. One could then even leave off of the map the administrative or political units that were used in the construction of the cartogram. And one could replace these by superimposing an alternate set of boundaries or other information (for example, disease incidence, poverty rates, roads, or shaded topography). This requires a bit of simple interpolation from the known point locations. It is also possible to lightly smooth the latitude and longitude graticule obtained to avoid inopportune kinks introduced by some algorithms.

It has recently been proposed that "brushing" techniques, borrowed from statistical graphics, can be used to overcome the problem of difficult geographical recognition on areal cartograms. In this procedure a normal map is presented alongside of the cartogram, and, by pointing at a location on one of the two maps, the comparable position on the other map is highlighted. Several implementations of this procedure can be found on the Internet. In this context, a point-wise inverse mapping is available. Of course the brushing technique only works on an interactive screen. To my knowledge, the efficacy of this method has not been studied.

As observed above, it is also possible to use Tissot's results to calculate the angular and linear distortion of the map. The areal distortion, Tissot's S, has been shown to be equal to the distribution being presented. Tissot's indicatrix (Robinson 1951; Laskowski 1989) is useful in

comparing two cartograms obtained from the same data, using alternative algorithms. If they both preserve the value-by-area property, then the measure of angular distortion (Tissot's ω) is important. The sum, or integral, of the indicatrix properties over the entire map can be used to provide an overall, that is, global, measure of the distortion. A similar result holds true of cartograms that stretch space using time or cost distance rather than areal exaggeration. Sen (1976) also presents some possible alternative measures of distortion.

It is now also apparent that one can calculate a cartogram to fit on a globe—a mapping of the sphere onto itself—but this has as yet not been done. It is not necessary to invoke a plane map at all. Examples might be to represent surface temperature or annual precipitation, constructing the globe gores in the usual fashion. From such a representational globe any conventional equal-area map projection (Eckert, Mollweide, sinusoidal, Lambert, etc.) can be used to represent the information as a proper (but different, depending on the initial projection) areal anamorphose on a flat map. Satellite image globes have to some extent supplemented political globes and anamorphic globes might someday also be constructed.

Extending the Concept

Computer scientists, including Edelsbrunner and Waupotisch (1997), have also studied the problem. In general they conceive of area cartograms as having a graphical display function—yielding insight into some problem—rather than as an analytical tool, and not a graphical nomograph for the solving of a specific problem. A recent implementation is from Texas, again in a thesis (Kocmoud 1997, also Kocmoud and House 1998). This interesting algorithm attempts to maintain shape in addition to proceeding to correct areal sizes. The shape preservation alternates with area adjustment in each iteration, neither being completely satisfied. Somewhat similar, but very considerably faster, programs and algorithms have recently been developed by researchers at the Martin-Luther University in Halle (Panse 2001) and the AT&T Shannon Research Laboratory (Keim, North, and Panse 2002; Keim, North, Panse, and Schneidewind 2002). These papers also review and display copies of existing alternative cartogram algorithms.

It is important to recognize that these computer engineers have employed a different objective. Instead of an areal cartogram, *sensu strictu*, they attempt to “balance” the value-by-area concept with shape preservation. That is, they allow departure from the objective of exactly fitting the areal distribution of concern to better conserve shapes. They introduce two finite measures of departure from the

objective. One applies to area, the other to shape. Contrast this with my objective of faithfully preserving the areal distribution and doing this in a manner that then tries to best preserve shape, without giving up the precise value-by-area property so necessary for the theoretical objective. This engineering approach is reminiscent of the Astronomer Royal George Airy's (1861) projection by a “balance of errors.” Airy, recognizing that a geographic map could not be equal area and conformal at the same time, chose to give up an exact fit to either of these important properties and instead used a least squares approach to find a compromise map projection. Aims comparable to those of Airy have been invoked for conventional map projections, with slightly different mathematical objective functions, by Jordan (1875), Kavraisky (1934), and others, and are described by Frolov (1961), Biernacki (1965), Mescheryakov (1965), and Canters (2002). Tissot's areal distortion (S) on a value-by-area cartogram (also equal to his a^*b , the product of the indicatrix axes) has been shown to be equal to $h(\varphi, \lambda)$, the areal density of concern, therefore a new criterion, comparable to that of Airy, can be formulated as a balancing of the value-by-area property with the property of conformality (local shape preservation). Then in order to use the least squares criterion, as did Airy, for this new objective we need to minimize the following double integral,

$$\int \int \left\{ [ab - h(\varphi, \lambda)]^2 + [a - b]^2 \right\} \cos \varphi \partial \varphi \partial \lambda$$

taken over the region of concern. Some variants of this integral are also possible—differential weighting of the criteria is an example. In the case of plane maps one substitutes u, v for the latitude and longitude and drops the cosine term. Observe that this is a global, not a local criterion, which instead would be used to minimize the maximum of the proposed function at all locations. The first squared term measures the departure from the value-by-area property. The second term measures the departure from conformality, or local shape preservation. The ratio of the minimum linear stretching to the maximum stretching (b/a), or the logarithm thereof, could also be used to measure departure from conformality, as is done in the closely related field of quasiconformal mapping (Teichmüller 1937; Gehring 1988). The recently invented computer algorithms use finite measures of the fit to the phenomenon of concern and to the preservation of shape. This is done by an alternating iteration until some combined criterion is satisfied. They then present very speedy, and recognizable, anamorphoses on which additional information may be displayed. Presumably, by varying the weight given angular or areal distortion, a range of solutions could be obtained. It would be of interest to see

whether new compromise (more conventional) map projections of the world, or parts thereof, can be created using these algorithms, simultaneously minimizing areal and shape distortion when given information defined by polygons. A map of the contiguous United States, using county areas defined by finite polygon coordinates, for example, or spherical quadrilaterals, might approximate, or balance, both Albers's equal-area conic projection and Lambert's conformal conic projection for this region. Interpolation could then produce a table of map projection coordinates for general use.

Along with Tissot's measure of areal distortion for evaluating a map projection, it is now possible to add a measure of the departure from density preservation, in addition to the conventional measures of linear and angular distortion. My initial objective was to have *no* departure from density preservation, combined with minimal departure from conformality. The newly introduced versions have departures from both criteria in a "balanced" fashion. Now the two independent measures of departure can be shown on the anamorphoses, using Tissot's indicatrix or by choropleths or isolines. As long as the cartograms are used only for visual display, it is clear that some departure from density preservation and from conformality can be tolerated since MacKay (1954, 1958) has shown that visual estimates of area and conformality are not terribly accurate.

Still Needed

As stressed in the foregoing materials, the general factors in evaluating an algorithm must be to (1) show correct value-by-area, (2) preserve shape to the extent possible, and (3) be efficient, in that order. For the last item the computational complexity of a cartogram would appear to be the same as that of any map projection. I am not aware of any results in this direction, but I would expect to find this to be polynomial in nature. The enhanced possibility of producing the inverse transformation also seems useful for theoretical purposes, using the important transform-solve-invert paradigm (Eves 1980, 215–28). This is similar to the use of a conformal Zhukovskii transformation in order to study the aerodynamics of airfoils (Ivanov and Trubetskov 1994, 70–73), and should also be useful in the medical use of cartograms. Sen (1976) also raises the question of uniqueness. In the case of "balanced" cartograms what metrics should be used for the tradeoff?

As a final remark, none of the several algorithms presented to date are capable of replicating Raisz's "rectangular statistical cartogram." This seems to be because the exact, correct topology is difficult or impos-

sible to maintain. The adjacency graph of the original unit outlines does not agree with that of the "rectangular" representation. This can probably be proven to be impossible, along topological lines similar to Euler's Königsberg bridge problem. Raisz actually forces his entire map (Figure 3) to fit into a rectangle, sometimes with an appended square for New England or Florida. In the last 35 years, several atlases have featured world cartograms with all countries as squares or rectangles, in their approximately correct position and size, according to some property, usually population or per capita income, and so forth (see Dent 1999, for citations). The countries are then colored according to some additional attribute. The "square-like" countries are not generally forced into an overall rectangle. The similarity is to Levasseur's map of Europe (Figure 2). To my knowledge, all of these are still produced by hand, perhaps assisted by a calculator for the arithmetic. An interactive computer program should be convenient for constructing this type of map. For the evaluation, a useful critical measure might be based on the distance of the true adjacency matrix from that as represented on the cartogram, but I am not aware of any such proposals. Only at the intersections of political boundaries, whose latitude and longitude are known, could anything like Tissot's indicatrix be calculated or could the entire spherical graticule be interpolated to illustrate the warping of the usual map. But perhaps other possibilities exist for measures for the measuring of the similarity of two cartograms, along the lines given in Sen (1976) or Tobler (1986b).

The computer construction of cartograms has progressed rapidly in the last several years. I expect that, with the increased speed and storage capabilities of future computers the next 35 years will lead to further changes in this field. As an illustration of this, an unpublished manuscript by two physicists (Gastner and Newman 2003) came to my attention, as the present paper was undergoing proofing. In this manuscript, they use the diffusion equation in the Fourier domain and allow variable resolution. This can be considered a mathematical version of Gillihan's (1927) smoothing procedure to compute a value-by-area cartogram.

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References

- Airy, G. 1861. Explanation of a projection by balance of errors for maps applying to a very large extent of the earth's surface, and comparison of this projection with other projections. *Philosophical Magazine* 22:409–21.
- Appel, A., C. Evangelisti, and A. Stein. 1983. Animating quantitative maps with cellular automata. *IBM Technical Discovery Bulletin* 26 (3A): 953–56.
- Biernacki, F. 1965. *Theory of representation of surfaces for surveyors and cartographers*, trans. from Polish, 53–147. Washington, DC: U.S. Department of Commerce.
- Brandon, C. 1978. Confrontations with cartograms: Computer-aided map designs and animations. B.A. thesis, Department of Visual and Environmental Studies, Harvard University.
- Canter, F. 2002. *Small-scale map projection design*. London: Taylor & Francis.
- Cauvin, C., and C. Schneider. 1989. Cartographic transformations and the piezopleth maps method. *Cartographic Journal* 26:96–104.
- Dent, B. 1999. *Cartography: Thematic map design*. 5th ed. New York: McGraw Hill.
- Dorling, D. 1991. *The visualization of spatial structure*. Thesis, Department of Geography, University of Newcastle-upon-Tyne, U.K.
- . 1995. *A new social atlas of Britain*. Chichester, U.K.: Wiley.
- . 1996. Area cartograms: Their use and creation. *Concepts and techniques in modern geography (CATMOG)*, No. 59.
- Dougenik, J., A. Nicholas, R. Chrisman, and D. Niemeyer. 1985. An algorithm to construct continuous area cartograms. *Professional Geographer* 37 (1): 75–81.
- Du, C. 1999. Constructing contiguous area cartogram using ArcView Avenue. *ESRI International Conference*, Paper No. 489. San Diego, CA.
- Eastman, J., W. Nelson, and G. Shields. 1981. Production considerations in isodensity mapping. *Cartographia* 18 (1): 24–30.
- Edelsbrunner, H., and R. Waupotisch. 1997. A combinatorial approach to cartograms. *Computational Geometry* 7: 343–60.
- Elsasser, H. 1970. Nichtflächenproportionale Kartogramartige Darstellungen der Schweiz. *Geographica Helvetica* 25 (2): 78–82.
- Eves, H. 1980. *Great moments in mathematics (before 1650)*. Providence, RI: The Mathematical Association of America.
- Fotheringham, A., C. Brunson, and M. Charlton. 2000. *Quantitative geography*. London: Sage.
- Friis, H. 1974. Statistical cartography in the United States prior to 1870 and the role of Joseph C. G. Kennedy and the U.S. Census Office. *The American Cartographer* 1 (2): 131–57.
- Frolov, Y. 1961. Issledovanie iskajenii ravnoelkich pseudo-tzilundaricheskich projektzii. *Vestnik. University Geologya Geogr* (12): 148–57.
- Funkhouser, H. 1937. Historical development of the graphical representation of statistical data. *Osiris* 3:269–404.
- Gastner, M., and M. Newman. 2003. Density equalizing maps for representing human data. Unpublished manuscript, Department of Physics, University of Michigan, Ann Arbor.
- Gehring, F. 1988. *Quasiconformal mappings*. Invitational presentation, 1986 International Congress of Mathematicians, 60 minutes, video, University of California, Berkeley.
- Gillihan, A. 1927. Population maps. *American Journal of Public Health* 17:316–19.
- Griffin, T. 1980. Cartographic transformation of the thematic map base. *Cartography* 11 (3): 163–74.
- . 1983. Recognition of area units on topological cartograms. *The American Cartographer* 10:17–28.
- Gusein-Zade, S., and V. Tikunov. 1993. A new method for constructing continuous cartograms. *Cartography and Geographic Information Systems* 20 (3): 167–73.
- Hägerstrand, T. 1957. Migration and area. In *Migration in Sweden*, Lund studies in geography, Series B, Human geography, No. 13, Department of Geography, University of Lund, Lund Sweden, 27–158.
- Hankins, T. 1999. Blood, dirt, and nomograms. *Isis* 90:50–80.
- Hunter, J., and J. Young. 1968. A technique for the construction of quantitative cartograms by physical accretion models. *Professional Geographer* 20 (6): 402–07.
- Inglis, K. 2001. *Computer generated cartograms*. Undergraduate thesis, Computer Science Department, Emmanuel College, Cambridge University.
- Ivanov, V., and M. Trubetskov. 1994. *Handbook of conformal mapping with computer-aided visualization*. Boca Raton, LA: CRC Press.
- Jackel, C. 1997. Using ArcView to create contiguous and noncontiguous area cartograms. *CAGIS* 24 (2): 101–09.
- Jordan, W. 1875. Zur Vergleichung der soldnerschen Koordinaten. *Zeitschrift für Vermessungswesen* 4:175–90.
- Kadman, N., and E. Shlomi. 1978. A polyfocal projection for statistic surfaces. *The Cartographic Journal* 15 (1): 36–41.
- Kavrayskiy, V. 1934. *Mathematischeskaya Kartografiya*, 229–36. Moscow: Redbaza Goskartotresta.
- Keim, D., S. North, and C. Panse. 2002. *CartoDraw: A fast algorithm for generation of contiguous cartograms*. Florham Park, NJ: AT&T Shannon Research Laboratory report.
- Keim, D., S. North, C. Panse, and J. Schneidewind. 2002. *Efficient cartograms: A comparison*. Florham Park, NJ: AT&T Shannon Research Laboratory report.
- Kocmoud, C. 1997. *Constructing continuous cartograms: A constraint-based approach*. M.A. thesis, Texas A&M University.
- Kocmoud, C., and D. House. 1998. A constraint-based approach to constructing continuous cartograms. Proceedings, 8th International Symposium on Spatial Data Handling, Vancouver.
- Kretschmar, M. 2000. *Visuelle exploration geographiebezogener Daten mit Hilfe von Kartogrammen*. Halle/Saale: Diplomarbeit.
- Kretschmer, I., J. Dörflinger, and F. Wawrik, eds. 1986. *Lexikon zur geschichte der Kartographie*. 2 vols. Wien: Deuticke.
- Lambert, J. 1992 [1772]. *Notes and comments on the construction of terrestrial and celestial maps, 1772*. Intro. and trans. W. Tobler. Ann Arbor: Geography Department, University of Michigan.
- Laskowski, P. 1989. The traditional and modern look at Tissot's indicatrix. In *Accuracy of spatial databases*, ed. M. Goodchild and S. Gopal, 155–74. London: Taylor & Francis.
- Levison, M., and W. Haddon. 1965. The area adjusted map: An epidemiological device. *Public Health Reports* 80 (1): 55–59.
- Lichtner, W. 1983. Computerunterstützte Verzerrung von Kartenbildern bei der Herstellung thematischer Karten. *Internationales Jahrbuch für Kartographie* 23:83–96.
- MacKay, J. R. 1954. Geographic cartography. *The Canadian Geographer* 4:1–14.
- . 1958. Conformality: Mathematical and visual. *The Professional Geographer (New Series)* 1 (5): 12–13.

- Mayet, P. 1905. Die schematischen-statistischen Karten des Kaiserlichen Statistischen Amtes zu Berlin. *Bulletin de l'Institut International de Statistique* 14, 3e livraison: 214–22.
- Merrill, D. 1998. Density equalizing map projections (cartograms) in public health applications. Doctoral dissertation, University of California School of Public Health. Also <http://library.lbl.gov/docs/LBNL/416/24/HTML>.
- Merrill, D., S. M. Selvin, and M. Mohr. 1991. Analyzing geographic clustered response. *Proceedings, 1991 Joint Statistical Meeting of the American Statistical Association*. Also Lawrence Berkeley Laboratory Report No. 30954, June; See the references at <http://www.library.lbl.gov/docs/LBNL/416/24/html/refs.html>.
- Meshcheryakov, G. 1965. The problem of choosing the most advantageous projections. *Geodesy and Aerophotography* 4:263–68.
- Monmonier, M. 1977. Nonlinear reprojection to reduce the congestion of symbols on thematic maps. *The Canadian Cartographer* 14 (1): 35–47.
- Olson, J. 1976. Noncontiguous area cartograms. *The Professional Geographer* 28 (4): 371–80.
- O'Sullivan, D., and D. Unwin. 2002. *Geographic information analysis*. Hoboken, NJ: Wiley.
- Palsky, G. 1996. *Des chiffres et des cartes: Naissance et développement de la cartographie quantitative française au XIXe siècle*. Paris: Comité des travaux historiques et scientifiques.
- Panse, C. 2001. *Visualisierung geographiebezogener daten*. Halle: Diplomarbeit, Martin-Luther Universität.
- Paullin, C. 1932. *Atlas of the historical geography of United States*. Washington, DC: Carnegie Institution and American Geographical Society.
- Raisz, E. 1934. The rectangular statistical cartogram. *Geographical Review* 24:292–96.
- . 1936. Rectangular statistical cartograms of the world. *Journal of Geography* 35:8–10.
- . 1938. 1948. *General cartography*. New York: McGraw Hill.
- . 1962. *Principles of cartography*. New York: McGraw Hill.
- Rase, W. 1997. Fischauge-Projektionen als kartographische Lupen. *Salzburger Geographische Materialien*, Heft 25, Department of Geography, University of Salzburg, Salzburg, Austria, 115–122.
- . 2001a. Kartographische Anamorphosen und andere nicht lineare Darstellungen. *Kartographische Bausteine*, 19, Institut für Kartographie, Technische Universität Dresden, Dresden, Germany.
- . 2001b. Volume preserving interpolation of a smooth surface from polygon-related data. *Journal of Geographical Systems* 3 (2): 199–219.
- Robinson, A. 1951. The use of deformational data in evaluating world map projections. *Annals of the Association of American Geographers* 41:58–74.
- . 1967. The thematic maps of Charles Joseph Minard. *Imago Mundi* 21:95–108.
- . 1982. *Early thematic mapping*. Chicago: University of Chicago Press.
- Ruston, G. 1971. Map transformations of point patterns: Central place patterns in areas of variable population density. *Papers and Proceedings of the Regional Science Association*, 28:111–29.
- Sarkar, M., and M. Brown. 1994. Graphical fisheye views. *Communications, Association of Computer Machinery* 37 (12): 73–84.
- Selvin, S., D. Merrill, J. Schulman, S. Sacks, L. Bedell, and L. Wong. 1988. Transformations of maps to investigate clusters of disease. *Social Science and Medicine* 26 (2): 215–21.
- Sen, A. 1975. A theorem related to cartograms. *American Mathematical Monthly* 82 (4): 382–85.
- . 1976. On a class of map transformations. *Geographical Analysis* 8 (1): 23–37.
- Skoda, L., and J. Robertson. 1972. Isodemographic map of Canada. *Geographical paper No. 50*. Department of Environment (En 36-506/50), Ottawa.
- Slocum, T. 1999. *Thematic cartography and visualization*. Upper Saddle River, NJ: Prentice Hall.
- Snyder, J. 1987. Magnifying glass azimuthal map projections. *The American Cartographer* 14 (1): 61–68.
- Teichmüller, O. 1937. Eine Anwendung quasikonformer Abbildungen auf das Typen Problem. *Deutsche Mathematik*, 2: 321–27. Reprinted 1982 in *Oswald Teichmüller: Gesammelte Abhandlungen*. (Collected papers) ed. L. Ahlfors and F. Gehring, Berlin: Springer.
- Tikunov, V. 1988. Anamorphated cartographic images: Historical outline and construction techniques. *Cartography* 17 (1): 1–8.
- Tissot, M. 1881. *Mémoire sur la représentation des surfaces et les projections des cartes géographiques*. Paris: Hermann et cie.
- Tobler, W. 1961. *Map transformations of geographic space*. Ph.D. Thesis, University of Washington, Seattle (University Microfilm #61-4011).
- . 1963. Geographic area and map projections. *The Geographical Review* 53:59–78.
- . 1972. Lambert's notes on maps, 1772. Introduced and translated. *Michigan Geographical Publication* (8). Ann Arbor: University of Michigan.
- . 1973. A continuous transformation useful for districting. *Annals, New York Academy of Science* 219:215–20.
- . 1974. Cartogram programs for geographic data given by latitude-longitude quadrilaterals, by irregularly shaped polygonal areas, and by a mathematical equation. *Cartographic Laboratory Report No. 3*, Department of Geography, Ann Arbor: University of Michigan.
- . 1979a. Cartograms and cartosplines. In *Automated cartography and epidemiology*, 53–58. Washington, DC: Department of Health, Education, and Welfare PHS-79-1254.
- . 1979b. Smooth pycnophylactic interpolation for geographic regions. *Journal of the American Statistical Association* 74 (367): 519–36.
- . 1984. Interactive construction of contiguous cartograms. *Computer Graphics '84*, 13–17. Anaheim, CA: National Computer Graphics Association Conference.
- . 1986a. Pseudo-cartograms. *The American Cartographer* 13 (1): 43–50.
- . 1986b. Measuring the similarity of map projections. *The American Cartographer* 13 (2): 135–39.
- Torguson, J. 1990. *Cartogram: A microcomputer program for interactive construction of contiguous value-by-area cartograms*. M.A. thesis, University of Georgia, Athens.
- Wallis, H., and A. Robinson, eds. 1987. *Cartographical innovations*. Tring, Herts, U.K.: International Cartographic Association.
- Wiechel, H. 1903. *Zwei Wahlkarten des Deutschen Reiches*, Gotha. (Cited in Kretschmer et al., 1986, 397).
- Williams, A. 1978. Interactive cartogram program on a microprocessor graphics system. *ACSM Fall Technical Meeting* 426–31.

Wright, J., L. Jones, L. Stone, and T. Birch. 1938. *Notes on statistical mapping, with special reference to the mapping of population phenomena*, 13, 14. New York: The American Geographical Society and the Population Association of America.

Yang, Q., J. Snyder, and W. Tobler. 2000. *Map projection transformation: Principles and applications*. London: Taylor & Francis.

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