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Majority Voting and the Location of Salutary Public Facilities

INTRODUCTION

For several decades, social scientists have been concerned with the systematic analysis of voting processes. Majority rule, which is widely institutionalized in Western societies, has been the subject of numerous analytical and critical studies. The fact that majority rule fails to generate an equilibrium social choice under certain preference configurations has long been known. In the absence of equilibrium, majority voting is subject to intransitivities, in which the eventual choice among three or more alternatives depends on the order upon which they are voted. Social choice theorists have therefore devoted considerable attention to the analysis of intransitivities in majority voting (Sen 1977).

Three recent papers have been concerned with majority voting in the location of salutary facilities, for which it is assumed that each consumer wishes to be located as close to the facility as possible. These papers have assumed spatially dispersed demand, but instead of locating a facility to minimize total distance, or to minimize the distance to the farthest point, they have assumed that the individuals requiring the facility will vote over alternative locations. Rushton, McLafferty, and Ghosh (1981) showed in a simulation that majority voting will in general not lead to the same solution as the more conventional criteria; furthermore, they showed that intransitivities will occur among triples of possible locations even when all voters share the same preference rules. Hansen and Thisse (1981) and Wendell and McKelvey (1981) also assumed that individuals would always prefer a closer location to a more distant one. Both concentrated on the conditions under which a location that would be preferred to all other locations by a majority of voters could be found. Hansen and Thisse called such a point a Condorcet point, in contrast to the Weber (minimum total distance) and Rawls (minimum maximum distance) points. Wendell and McKelvey showed the isomorphism between voting and the behavior of an entrepreneur seeking a location that would be closer than any other location to a majority of consumers, and they cast their analysis as a search for competitive equilibrium. They distinguished

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between a set of locations giving strong equilibrium (G_s) and a set giving only weak equilibrium (G_w): for strong equilibrium a location must be preferred to all other potential locations by a strict majority, whereas weak equilibrium exists when the preferred location defeats or ties all alternatives.

Both Hansen and Thisse (1981) and Wendell and McKelvey (1981) were concerned with the existence of the Condorcet point, or the emptiness or nonemptiness of the sets G_s and G_w. Hansen and Thisse showed that a Condorcet point always exists on a tree network. Wendell and McKelvey showed that G_s and G_w will be nonempty only under certain conditions of symmetry, which can be stated both for network (discrete space) and continuous space problems. In the continuous space case, the conditions can be stated as follows. If the number of voters is even, then it must be possible to find a pairing of all individuals such that when their locations are connected by straight lines, all lines pass through a common point that lies between the two individuals in each pair. If the number is odd, the unpaired individual must be at the intersection. In summary, it seems that given the general conditions likely to exist in real-world problems, both G_s and G_w can be expected to be empty. Wendell and McKelvey see this situation as "rather bleak." Hansen and Thisse suggest that it would be logical to move to criteria that are less demanding than the strict majority (Kramer 1973).

Given the improbability of a simple solution, the purpose of the present paper is to examine some of the more general aspects of the Condorcet location problem. Of particular concern are the potential conditions under which one location is preferred to another by majority rule. These conditions are used to determine those triples of locations that give rise to intransitivity. Results are obtained in continuous Euclidean space, first for the traditional Weber triangle and then for the general case of N weighted demand points.

UNWEIGHTED TRIANGLE

Consider first the case of three unweighted points. A Condorcet point must exist only if the three points are collinear (Black 1948). However, when the three points are not collinear, equilibrium exists only under the conditions developed below. The general problem is illustrated in Figure 1. If point P is one of two potential facility sites, a second site Q will defeat P iff Q is located within the shaded portions of Figure 1a, defined by arcs of circles drawn centered on the three points; otherwise P will defeat Q. The size of the shaded arc varies systematically with the location of P, but would be zero only if P were a Condorcet point. In general it is impossible to find a P that cannot be defeated by some Q.

The area that dominates P within the triangle is shown in Figure 1a by the symbols a, b, and c. Let R_A be the area of the segment of a circle of radius AP centered on A and defined by the sides AB and AC. Then

$$R_A = b + c + d \tag{1}$$

$$R_B = c + a + e \tag{2}$$

$$R_C = a + b + f . \tag{3}$$

Summing, we have an expression for the shaded area within the triangle:

$$S = R_A + R_B + R_C . \tag{4}$$

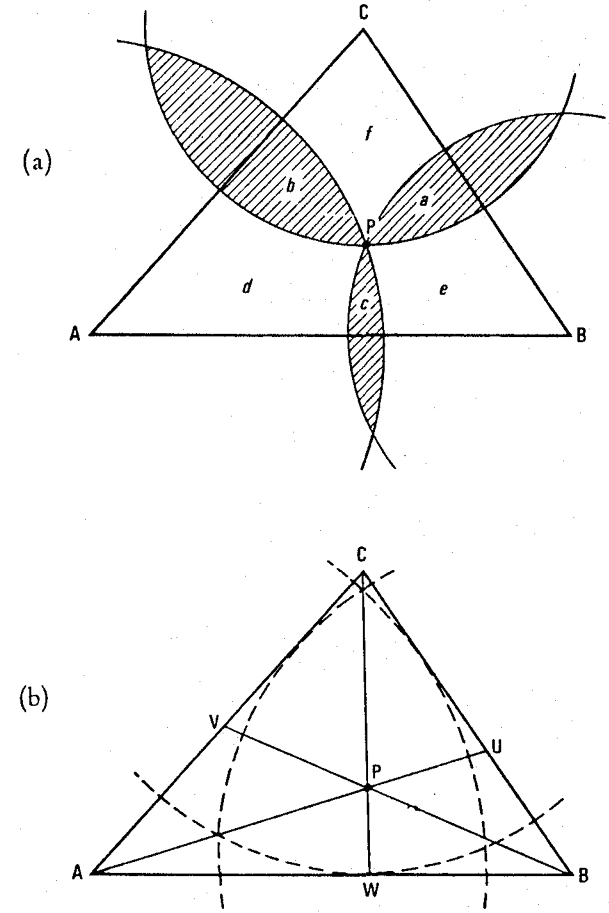

FIG. 1. Definitions of Terms

Since the sides of the triangle bisect each of the three "leaves" of the shaded area in Figure 1a, the total shaded area is given by

$$S = 2(R_A + R_B + R_C - \Delta ABC) . \qquad (5)$$

If, on the other hand, P lies within triangle ABC but outside the dashed lines shown in Figure 1b, which are arcs of circles centered on a vertex and tangential to the opposite edge, then less than half of the appropriate leaf will lie inside the triangle. However, it is easy to show that as long as P lies inside the triangle, the expression for S given in equation (5) is still valid.

We might now ask where P should be located within the triangle so as to minimize the area of dominance S. In other words, given that no Condorcet point, for which $S = 0$, exists, which is the location that is least subject to defeat as measured by the area of domination? Let (P_1, P_2), (a_1, a_2), (b_1, b_2), and (c_1, c_2) be the coordinates of points P, A, B, and C respectively. Then, the minimum point can be found by defining the components of equation (5) in terms of these coordinates using the Pythagorean theorem and the familiar geometric conditions characteristic of triangles. Taking the derivatives of S with respect to P_1 and P_2 show that a minimum occurs at

$$P_1 = (a_1\angle_A + b_1\angle_B + c_1\angle_C)/\pi \tag{6}$$

$$P_2 = (a_2\angle_A + b_2\angle_B + c_2\angle_C)/\pi \, , \tag{7}$$

where $\angle_A$, $\angle_B$, and $\angle_C$ are the interior angles of the triangle.

Denote the location of the minimum value of S, whose coordinates are defined by equations (6) and (7), by $\hat{P}$. The properties of $\hat{P}$ can be seen with reference to Figure 1b. It can be shown that $\hat{P}$ divides CW in the ratio

$$\frac{W\hat{P}}{\hat{P}C} = \frac{\angle_C}{\pi - \angle_C} \, . \tag{8}$$

Similarly,

$$\frac{V\hat{P}}{\hat{P}B} = \frac{\angle_B}{\pi - \angle_B} \tag{9}$$

and

$$\frac{U\hat{P}}{\hat{P}A} = \frac{\angle_A}{\pi - \angle_A} \, . \tag{10}$$

Furthermore, $AW/WB = \angle_B/\angle_A$, $BU/UC = \angle_C/\angle_B$, and $CV/VA = \angle_A/\angle_C$. From these conditions, $\hat{P}$ always lies strictly within the triangle unless at least one of $\angle_A$, $\angle_B$, or $\angle_C$ is zero. $\hat{P}$ clearly coincides with the Condorcet point when it exists, but it coincides with the Weber point only in the special case of an equilateral triangle. Further contrasts between the behavior of $\hat{P}$ and the Weber point are indicated by comparison with the results obtained by Tellier (1972). To compare the properties of $\hat{P}$ with that of the centroid, which minimizes the sum $AP^2 + BP^2 + CP^2$, note that the centroid divides AU in the ratio of 2:1, and similarly for BV and CW.

The special case of an isosceles triangle illustrates the divergence between the Condorcet and Weber points and the different properties associated with each point. Substituting the characteristics of an isosceles triangle into equations (6) and (7) as before, the relationship between P_2, or the height of $\hat{P}$ above the base, and the height of triangle ABC, or CW, is given by

$$P_2 = [\pi - 2\tan^{-1}(CW/AW)]CW/\pi \, . \tag{11}$$

In the range $0 < CW < \infty$, this is a monotonically increasing function. By contrast, for the Weber point, $P_2 = AW/\sqrt{3}$. This indicates that for the Weber point, P_2 is independent of CW. For the centroid, $P_2 = CW/3$.

WEIGHTED TRIANGLE

Now consider the case of the three points A, B, and C, with weights w_1, w_2, and w_3 respectively. The critical areas are still those shown by shading in Figure

1a. However, the conditions for dominance in the six parts of the triangle are now as follows. P dominates points in a if $w_1 > w_2 + w_3$, and similarly for b and c; P dominates points in d if $w_1 < w_2 + w_3$, and similarly for e and f. Note that if equalities are considered, it is possible to distinguish weak dominance in which P is at least as preferred as Q. For example, P would be weakly dominant over a Q located in a if $w_1 = w_2 + w_3$. Since at most one of the first three conditions can be true, it follows that at most one of a, b, and c can be dominated, and that at least two of d, e, and f must be dominated. Suppose that $w_1 = w_2 + w_3$, and S is given by $b + c + d$. Clearly, S is minimized in this case by location at A. If the weights are distributed such that one point has more than half the total weight, a Condorcet point exists and P is at that point. However, if no point has more than half of the total, the solution is as in the case of unweighted points and is unaffected by the values of w_1, w_2, and w_3.

THE GENERAL CASE

This section considers the general case of N weighted points. Since the perpendicular bisector of the line segment connecting two points P and Q divides places closer to P from those closer to Q, it follows that P dominates Q iff the majority of the weight is found on P's side of the perpendicular bisector of PQ. It follows that for P to dominate all other potential locations and be a Condorcet point, any line drawn through P must divide the weight in that way. Wendell and McKelvey (1981) use the term "total median" to describe a point with this property.

An example of the area dominating a location P is shown in Figure 2, which is based on five equally weighted points. The area of dominance consists of the intersection of circles centered on each point and passing through P. When an intersection occurs between circles centered about a set of points whose combined weights exceed half of the total weight, locations within this intersection all dominate P. When one point has more than half of the total weight, the area of dominance will be the circle centered on that point.

Now let P, Q, and R be a triple of alternative locations. Intransitivity will occur when P dominates Q, Q dominates R, and R dominates P, or the reverse. The conditions that lead to intransitivity depend on the distribution of weights relative to the perpendicular bisectors of PQ, QR, and RP. Consider the triangle PQR illustrated in Figure 3. Since PQ, QR, and RP are all chords of the circle circum-

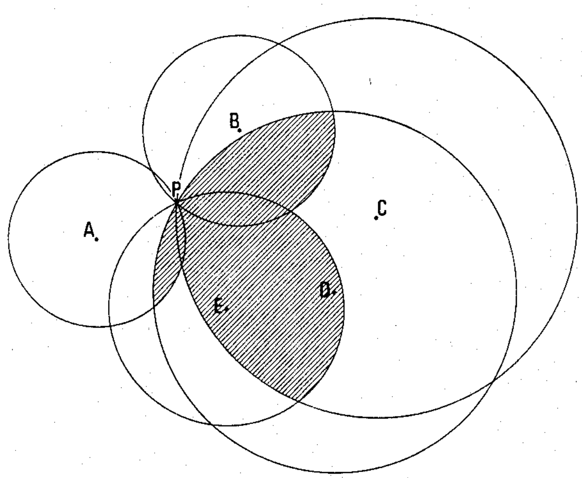

FIG. 2. Example of Dominance in the General Case

scribing the triangle, the three bisectors must meet at a point, labeled T, which is also the center of the circle, so that $TP = TQ = TR$. Let the plane be divided into six sectors defined about T by the three bisectors and labeled a through f, as shown in Figure 3. Let the symbols a through f denote the total weight of the points in their respective sectors. Then, the necessary conditions for intransitivity are

$$P \text{ dominates } Q \text{ implies that } f + a + b > c + d + e \tag{12}$$

$$Q \text{ dominates } R \text{ implies that } b + c + d > e + f + a \tag{13}$$

$$R \text{ dominates } P \text{ implies that } d + e + f > a + b + c\,. \tag{14}$$

Of course, the necessary conditions also hold if all three inequalities are reversed simultaneously.

Substituting $e = 0$ and $f = 0$ into equations (12) through (14) leads to the conclusion that the triple of points must be transitive if two or more adjacent sectors have zero weight. It follows that any triangle whose center T lies outside the convex hull of weighted points must be transitive, because construction of the perpendicular bisectors must yield two or more adjacent sectors with zero weight.

The conditions described in equations (12) through (14) suggest that intransitivity is a fairly common occurrence. To gain a general impression of the likelihood of intransitivity, a simulation experiment was carried out. The conditions reduce to

$$u > v + w \tag{15}$$

$$v > w + u \tag{16}$$

$$w > u + v \tag{17}$$

or the reverse, where $u = a - d$, $v = c - f$, and $w = e - b$. If T is a random point in an infinite plane, with weight uniformly distributed over this plane, then u, v, and w will be independent random variables with zero mean. In the simulation, u, v, and w were sampled from the Gaussian distribution with zero mean and unit standard deviation. In 10,000 trials, a total of 887, or 8.87 percent of the cases, were found to be intransitive.

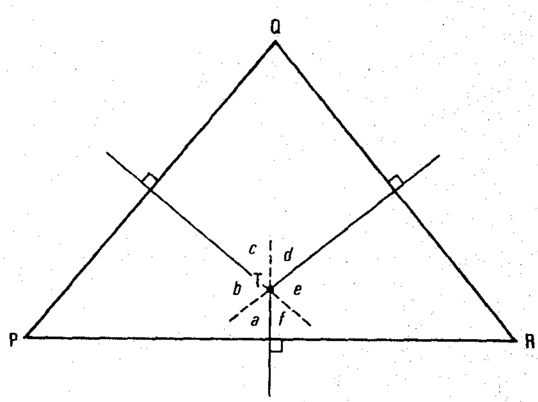

FIG. 3. Conditions for Intransitivity

CONCLUSIONS

This paper developed general conditions for the intransitivity of triples of potential salutary facility locations. The results indicate the distinctions between the Condorcet point, when it exists, and the Weber and Rawls points determined through the application of conventional locational objectives. When the Condorcet point does not exist, the point most likely to dominate all alternatives can be found; this point also diverges from the Weber and Rawls points. In the triangle, this point has simple properties, which provide an interesting contrast to those of the better-known measures of spatial centrality. An avenue for future work would be to devise an algorithm for finding this point in the general case.

The conditions for intransitivity between three alternative locations reduce to simple geometrical inequalities, and they suggest that intransitivity may be relatively common. A simulation revealed that nearly 10 percent of randomly selected cases generated resulted in intransitivity. Further research should be devoted to consideration of intransitivities in other cases.

LITERATURE CITED

Black, D. (1948). "On the Rationale of Group Decision Making." *Journal of Political Economy,* 56, 23–34.

Hansen, P., and J.-F. Thisse (1981). "Outcomes of Voting and Planning: Condorcet, Weber, and Rawls Locations." *Journal of Public Economics,* 16, 1–15.

Kramer, G. (1973). "On a Class of Equilibrium Conditions for Majority Rule." *Econometrica,* 41, 255–97.

Rushton, G., S. L. McLafferty, and A. Ghosh (1981). "Optimum Locations for Public Services: Individual Preferences and Social Choice." *Geographical Analysis,* 13, 196–202.

Sen, A. (1977). "Social Choice Theory: A Re-Examination." *Econometrica,* 45, 53–89.

Tellier, L.-N. (1972). "The Weber Problem: Solution and Interpretation." *Geographical Analysis,* 4, 215–33.

Wendell, R. E., and R. D. McKelvey (1981). "New Perspectives in Competitive Location Theory." *European Journal of Operational Research,* 6, 174–82.