THE FRACTIONAL BROWNIAN PROCESS AS A TERRAIN SIMULATION MODEL

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ABSTRACT

A stochastic model applicable to a wide variety of terrains would be useful for a number of reasons. The fractional Brownian processes appear to provide such a model. This paper reviews existing work in this area and makes an extensive empirical test of the model. It concludes that while departures from the model can be expected in almost any real terrain, these do not necessarily detract from its usefulness.

INTRODUCTION

A two-dimensional stochastic process capable of simulating a surface with a reasonable resemblance to real terrain would be useful for a number of reasons. First, a number of activities including topographic mapping and remote sensing rely on samples of terrain, from which images or statistics can be generated. The sampling intensity clearly affects the usefulness of the product, and a stochastic model of terrain would allow these effects to be quantified, so that objective decisions could be made about sample design. Second, such a stochastic process would provide a null hypothesis to which real terrain could be compared, in order to isolate aspects of the terrain which result from systematic, geomorphic processes. Third, such a process would provide a starting point for simulations of geomorphic processes, including the action of water, wind, ice and waves. Fourth, the problem of terrain simulation has generated some interest in the area of low-altitude flying (7), with applications in the defense industry. And finally, Dutton (2) has shown how terrain simulation can be useful in cartographic generalization and its inverse.

Mandelbrot (6) has published some striking illustrations of the ability of fractional Brownian processes to resemble certain types of real terrain. The main purpose of this paper is to make an empirical comparison between this class of stochastic models and a piece of real terrain, but first it would be useful to review some of their important properties.

THE FRACTIONAL BROWNIAN PROCESS

Let $z(\underset{\sim}{x})$ and $z(\underset{\sim}{x} + \underset{\sim}{d})$ represent the elevation at two points, $\underset{\sim}{x}$ and $\underset{\sim}{x} + \underset{\sim}{d}$ respectively, separated by a distance $|\underset{\sim}{d}|$. Then consider the expected squared difference between their elevations. On a fractional Brownian surface the expected value is a function of distance alone,

$$E[z(\underset{\sim}{x}) - z(\underset{\sim}{x} + \underset{\sim}{d})]^2 = |\underset{\sim}{d}|^{2H} \qquad (1)$$

where H is a parameter. The range of H is from 0 to 1: when it is large, the variability of the surface is small locally, but rises rapidly with distance, whereas when H is small the surface shows high local variability but a slow increase at large distances. Fractional Brownian surfaces can be simulated by a process described by Mandelbrot (5). Figure 1 shows a series of sample surfaces ranging from H = 0.5 to H = 0.8. Note that the particular features of each surface, including any general trend, location of peaks, pits and so on, are all outcomes of random generation.

Fractional Brownian surfaces are self-similar, so that any part of the surface, when scaled appropriately, is indistinguishable from the whole. If landscape were the outcome of such a process, this would imply that an observer examining any part of it would be unable to determine the scale. Since there undoubtedly are visual clues in the physical landscape which allow the observer to determine scale, at least within certain limits, it is clear

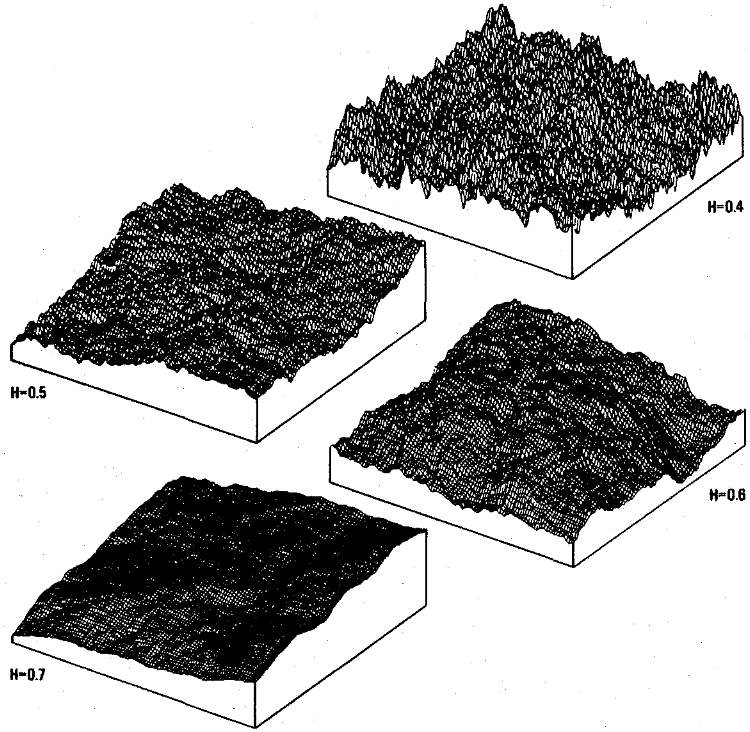

Figure 1. Self-similar surfaces generated by fractional Brownian processes.

that pure self-similarity is generally not a property of real landscape. However these clues are usually ascribable to geologic controls or the influence of geomorphic processes, and can be viewed as deterministic factors which have been imposed on the null hypothesis landscape. The landscapes which most resemble those in Figure 1, and which are most self-similar, tend to be those most lacking in geologic control or modification by geomorphic process, such as lunar or dead-ice topography, or landscapes dominated by recent tectonic activity.

The underside of a fractional Brownian surface is indistinguishable from the upper side, and pits and peaks should therefore occur with equal probability. A lake-rich glaciated topography is more likely to resemble the fractional Brownian than one which has been heavily modified by fluvial erosion or mass movement, which may be identifiable as deviations from the null hypothesis.

The feature which makes fractional Brownian processes most attractive as landscape null hypotheses is the behaviour of measures with scale. From the self-similarity property the configuration of any contour, cross section or element of surface is dependent on the scale of observation: in the limit each is infinitely complex. Contours and surfaces therefore lack tangents, and such measures as length or area are dependent on a defined scale of measurement, in a way which is at variance with the behaviour of smooth geometric curves but in agreement with the behaviour of such geographical features as coastlines (for review see (3), and see (9) for a further application).

The behaviour of measures with scale is characterized by a parameter D, the fractional dimension. On a fractional Brownian surface the length of a contour or coastline L is a function of the scale of measurement λ, $L(\lambda) \propto \lambda^{1-D}$, where for example λ might be the spacing of a pair of dividers used to "walk" along the contour, and $L = N\lambda$ the length estimate obtained by multiplying λ by the number of steps taken. Thus when log L is plotted against log λ a straight line is obtained with a slope of 1-D. For contours of the surface, D = 2-H. Similarly if a grid is laid over the map and those cells whose centres fall on land coloured black and those on water, white, and the length of the coastline estimated by counting the number of black-white joins, a plot of the log of the length estimate against the log of cell size will similarly yield a straight line with a slope of 1-D. Note that if the surface is smooth and H=1, then D=1 and length is independent of λ or cell size, and the contour or coastline behaves as a smooth curve with defined tangents.

EMPIRICAL TEST: RANDOM ISLAND, NEWFOUNDLAND

Despite the interest generated by Mandelbrot's illustrations, no strictly empirical test of the model has yet been published. Scheidegger (8) rejected the model in principle on the argument that self-similarity is not widely observed in real terrain, but this view seems overly rigid. Burrough (1) argued that many real phenomena display self-similarity over limited ranges of scale in a paper which estimated D as a parameter for a very wide range of phenomena, largely from spectra and variograms. The tests made in this paper are much more extensive, and are interpreted in the context of the geomorphic processes believed to have been responsible for the present form of the physical landscape. The test area is Random Island, which lies just off the coast of eastern Newfoundland.

Data were taken from the 1:50,000 Random Island sheet (NTS 2 C/4 printed 1972). The shoreline, 250' and 500' contours and outlines of lakes were digitized to an accuracy of .25 mm on a Tektronix 4954 tablet. All lakes of area greater than 0.05 cm^2 were included, and all contour loops satisfying the same criterion, to give a total of 153 lakes, 68 loops at 500', 22 at 250' and 5 at the shoreline, and two islands within lakes. A total of 8358 coordinate pairs were captured. Figure 2 shows the result. Each line was labelled with descriptions of the space on either side and stored in the arc data structure defined by PLUSX (4).

LENGTH MEASURES

The coastline, 250 and 500 foot contours and lake outlines were treated as four different sets of lines. Lengths were measured by simulating the stepping of dividers using a variety of settings from 1 mm to 4 cm, or from 50 m to 2 km. Length estimates were then totalled within each of the four groups, and plotted against step size.

The shoreline, 250' and 500' contours show strong linearity, with some deterioration at small step sizes because at these scales the accuracy of the original digitizing begins to affect the results. D values were estimated from the slopes, at 1.11, 1.19 and 1.31 respectively. The lake outlines result is less linear. There is the same reduction in slope for small step sizes, but an additional flattening at large step sizes since many of the small lakes disappear completely at these scales. In between the slope approaches -0.8, corresponding to D = 1.8. Although the data does not allow a reliable estimate of D for

Figure 2. Plot of shoreline, 250' and 500' contours and lakes, Random Island, Newfoundland.

the lakes, the results are at least consistent with the estimates obtained from other methods below (Table I).

Grid cells were overlaid on each of the four line sets, using square cells with sides ranging from 0.7 mm to 7 cm. Line lengths were then estimated by counting the number of joins between black and white cells, using the 4-neighbour or Rook's case. The logs of the join counts were then regressed against the log of cell size, and the slopes used to estimate D.

In all four graphs there was some deterioration of linearity at the 0.7 mm cell size, presumably because of the effects of digitizer accuracy. To allow for this, regressions were carried out both with (7 cases) and without (6 cases) this cell size, and the results, including the correlation coefficients, are shown in Table I. They show the same monotonic increase from the shoreline through to the lakes, but the numerical values are in most cases higher than with the estimates from the dividers.

LOOP ALLOMETRY

A further estimate of D can be made by comparison of the various loops in each of the line sets. Consider the lake outlines for example. If a single lake is reproduced at half scale its perimeter can be expected to halve, while its area will be reduced by a factor of four. In reality, however, each lake's outline will be drawn with a similar level of generalization determined by the scale of the map, so that if two lakes can be found with areas in the ratio 1:4, the second can be expected to have a perimeter rather more than twice as long as that of the first. Area A and perimeter P can be expected to follow a power law

$$A = k\,P^{b},\quad 1 < b < 2 \qquad (2)$$

over the set of observed lakes. At one extreme with smooth lake outlines b will approach 2, and at the other b=1 when the outlines are so complex as to completely fill each lake, so that perimeter behaves as a measure of area. In fact b = 2/D, and an estimate of D can therefore be obtained by regressing the log of loop area against the log of perimeter for each of the four line classes. Correlations were strong although the shoreline set had a sample size of only five loops, and are shown in Table I along with the estimates of D. Once again they show the same monotonic increase, but numerical values which are inconsistent with other estimates.

TABLE I

Estimates of D for four line sets, with associated correlations

	Shoreline	250 ft. contour	500 ft. contour	lake outlines
Dividers	1.11	1.19	1.31	--
Join counting, 7 cases	1.11 (-.9998)	1.29 (-.9990)	1.47 (-.9971)	1.53 (-.9959)
Join counting, 6 cases	1.13 (-.9998)	1.33 (-.9994)	1.54 (-.9984)	1.61 (-.9975)
Area/perimeter	1.14 (.9973)	1.17 (.9945)	1.19 (.9870)	1.30 (.9641)

DISCUSSION

In each of the previous analyses the Random Island topography exhibited the same type of behaviour as the stochastic process or null hypothesis, that is, a log-log relationship between length measures and scale, and the predicted form of loop allometry. However it is clear that Random Island differs from H_0 in a number of significant ways. First, estimates of D vary consistently for the four line sets. It would appear that the topography at the shoreline is considerably smoother (lower D) than in the centre of the island, since the 500 foot contour and lake outlines give a much higher D. All of the shorelines except the eastern are on the sides of flooded U-shaped valleys formed by glaciation and affected by some degree of structural control. In addition the shorelines have been modified by coastal processes, which tend to produce smoothing through the formation of beaches, spits, etc. It seems that the null hypothesis of a homogeneous fractional Brownian process must be rejected, but that one can conceive instead of a D which varies spatially from a low at the shoreline to a high in the centre, and from a low at the western end of the island to a high at the more rugged eastern end.

The length/scale relationships are highly linear, suggesting that the concept of self-similarity can be accepted at least over the range of scales used, but with some exceptions. The relationships tend to break down at scales below 2 mm, probably because of the limited accuracy of the digitizing process, but possibly because of cartographic generalization or the topography itself. For the lakes the dividers technique breaks down at scales above

2.5 cm, but this seems to be a problem of the method since the join count results are not affected. So although one would expect geomorphic and geologic controls to produce significant departures from self-similarity at certain scales, there is no clear evidence that this has happened on Random Island over the range of scales used in this analysis. Finally although the trends are consistent for each method, the numerical results obtained vary substantially, joint counting producing consistently higher estimates of D particularly for the 500 ft. contour and lake outline sets.

Fractional Brownian surfaces were defined earlier in terms of the relationship between expected squared differences and distance, which shows a continuous monotonic increase. This is reflected in the examples shown in Figure 1 in the presence of a "drift" or general trend of elevations, particularly for the smoother surfaces of high H. The further two points are apart, the more different their elevations. This is clearly not true of the Random Island topography, since pairs of points at opposite ends of the island are almost as similar in elevation as pairs within a mile of each other. Instead of rising indefinitely, the expected squared differences tend to be asymptotic to some maximum value which is achieved over a distance of a few miles. Thus while the local variability may exhibit the null hypothesis behaviour, as revealed by the length/scale relationships for example, long-distance variability does not. It may also affect the results of the join counting analysis, and may explain the inconsistencies noted in the previous section, although this must be regarded as merely speculative given the lack of more detailed data.

CONCLUSIONS

The fractional Brownian process has had mixed success in predicting aspects of the Random Island topography. It correctly and uniquely predicts the relationships between line lengths and scale and between loop area and perimeter, but the parameters of these relationships are numerically inconsistent. The parameters vary in a manner which suggests the existence of a nonhomogeneous fractional Brownian process, with systematic spatial variation in the H or D parameter according to elevation and position. These variations are understandable in terms of the geomorphic history of the island. Within the range of scales of these analyses there is good adherence to the concept of self-similarity.

We cannot, therefore, simulate Random Island by a single run of a fractional Brownian process with specified H, because of the spatial nonhomogeneity of the island and the relative lack of long-range variability. But the class of fractional Brownian processes is clearly able to model many of the features of the topography, and H and D provide useful spatial indices. None of these processes, however, would produce a landscape indistinguishable from Random Island in general properties to the extent that a physical geographer would accept it as a new piece of the coast of Newfoundland.

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