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Topologic Model for Drainage Networks With Lakes

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Shreve's probabilistic-topologic model for drainage network topology is herein extended and generalized to allow for the presence of lakes. Drainage network topology is represented by an integer string directly analogous to the binary strings used for channel networks without lakes. Validity constraints on integer strings are presented, along with combinatorial results and methods for generating 'topologically random' networks. The hypothesis that network element degree and type is independent of position within the integer string leads to good predictions of the relative frequencies of various classes of small subnetworks within a 596-link network in northern Ontario. For the special case of networks without lakes the model is equivalent to Shreve's.

INTRODUCTION

R. L. Shreve's 'probabilistic-topologic model' for channel networks [*Shreve*, 1966, 1967, 1969, 1974, 1975] has proven to be one of the most successful quantitative models in landscape-scale geomorphology. While the model fails to predict certain local properties of networks (for example, see *James and Krumbein* [1969]), it accounts for many of the overall properties of drainage networks and drainage basins, from 'Horton's Laws' to the 'mainstream-length:basin-area' relation [see *Shreve*, 1975; *Smart and Werner*, 1976; *Jarvis*, 1977; *Smart*, 1978].

One weakness of Shreve's model is that it cannot directly be applied to networks containing 'lakes fed by multiple inlets' [*Shreve*, 1966, p. 20], a feature shared by *Horton*'s [1945] (see also *Strahler* [1952]) pioneering work. That this has not been considered a major flaw is indicated by the fact that almost all papers on network topology have ignored lakes altogether; Shreve himself is one of the few to mention them, and he simply assumes that lakes are not present in the networks studied [*Shreve*, 1966, 1967, 1969]. However, lakes are prominent features of most glaciated landscapes, landscapes which cover some 30% of the earth's present land area. Sample counts on the Canadian Shield, at the resolution of 1:50,000 scale maps, have yielded typical lake densities of 0.4–0.6 lakes per square kilometer, which would indicate some 2 million lakes in that physiographic province alone.

In this paper we present a more general topologic model for stream networks which allows for the possibility of lakes with multiple inlets. An important feature of our model is that it collapses to Shreve's when the number of lakes equals zero. The model networks can be represented by integer strings directly analogous to the binary strings [*Shreve*, 1967] used for trivalent networks. We discuss the properties of these strings, some combinatorial results for them, and algorithms for generating 'random' networks within a certain class. We conclude with a preliminary test of an extension of *Shreve*'s [1967] 'topological randomness' hypothesis to networks with lakes.

THE MODEL

A lake is defined as a body of surface water, not a part of the world ocean, in which water flow velocities are too low to transport suspended sediment. This definition leads occasionally to difficulties in identifying individual lakes on maps (How much must a river widen before a lake is considered to be present? How small must a 'narrows' be before a lake is subdivided into two lakes?). The stream network is then considered to be a 'planted plane tree' with two classes of vertices: 'normal points,' which must be of degree 1 (point sources plus the basin outlet) or 3 (stream junctions), and 'lakes,' which may be of any degree greater than or equal to 1.

In graph theory, two graphs are said to be isomorphic if there exists a one-to-one correspondence of their vertices which preserves adjacencies. *Shreve* [1966, p. 27] introduced the term 'topologically identical' for isomorphic channel networks which could be made to correspond exactly by continuous deformation within the plane. The complement of this term is 'topologically distinct,' and the abbreviation TDCN has become standard for 'topologically distinct channel networks.' Channel networks which are isomorphic but not necessarily topologically distinct are said to belong to the same ambilateral class [*Smart*, 1969]. Extending these concepts, we require in addition a correspondence of vertex type (lake or normal point) for isomorphism. Systems possibly containing lakes which cannot be made to correspond by continuous deformation within the plane are termed 'topologically distinct drainage networks' (TDDN). The term 'ambilateral class' is retained without change.

We assume that the tree is a directed graph pointing toward the outlet and that each lake is of out-degree exactly 1. Although lakes with more than one natural outlet do exist (for example, Lake Anima-Nipissing in the Montreal River basin in the Temiskaming area of northeastern Ontario), they are very rare. We also exclude at present lakes with no surface outlets, and we also ignore islands within channels. Furthermore, our model cannot accommodate drainage systems on islands in the lakes within the network. The fact that our model cannot accommodate these situations is not considered to be an important disadvantage. In schematic diagrams of network topology we represent lakes by open circles of constant size, regardless of the lakes' true sizes and shapes; an example is shown in Figure 1.

Integer String Representation

Scheidegger [1967] and *Shreve* [1967] reported that the topology of a trivalent planted plane tree can be represented by a binary string. *Smart and Werner* [1973] described a number of algorithms for generating and processing these

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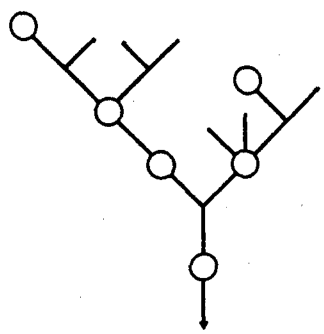

S = {3034021011511021}

Fig. 1. Schematic representation of a drainage network including lakes (circles) and S, the integer string representation of the network (see text).

strings. Each link in the network is represented in the string by a binary digit indicating the type of vertex at its upstream end: 0 if that is a junction (interior link) or 1 if that is a source (exterior link). The string begins at the outlet and thus must start with a zero (unless the network has only one link). The links are traversed in order by turning left at each junction and reversing at each source, and the code for link type is recorded only the first time that a particular link is traversed. For a trivalent network with μ sources there are $2\mu - 1$ links and thus $2\mu - 1$ elements (μ ones and $\mu - 1$ zeros) in the binary string which uniquely represents the network topology. The number of sources (μ) is termed the magnitude of the network. Link properties (lengths, areas, gradients, etc.) may be readily manipulated if stored in 'parallel' vectors, also of $2\mu - 1$ elements.

For planted plane trees with V vertices each of unrestricted degree, the topology can be represented uniquely by a string of $2V$ binary elements [*deBruijn and Morselt,* 1967]. Beginning at the roof of the graphs, links are coded U as they are traversed in the upstream direction and D in the downstream. However, in the present case, point sources and lakes with no inlets, while both vertices of degree 1, are considered to be geomorphologically and hydrologically distinct; similarly, lakes with two inlets must be distinguished from forks. By adding a binary digit after each U to indicate whether or not the link drains a lake, such networks could be represented by $3V$ elements. While this is a very compact representation, we propose an alternative coding system which has many advantages in the processing of stream networks.

A stream network, containing lakes, can be represented by a string S of k integers, one for each of the k channel links in the network. Each link's code integer depends again on the vertex at its upstream end: normal junctions are represented by 0, normal sources by 1, and lakes by integers s_j, where s_j equals 2 more than the number of lake inlets (or, 1 more than the degree of the vertex). As in the binary strings used for trivalent networks, links are traversed beginning at the basin outlet and turning left at each junction. If a link drains a lake, the next link in sequence is the first inlet to the lake encountered as one moves clockwise around the lake shore; if the lake has no inlets, one reverses back down the last link, as one does at normal sources in the binary string. Also as with trivalent networks, links are only recorded the first time they are traversed. This integer string representation uniquely defines the topology of a stream network with lakes. An example is given in Figure 1. Note that the particular codes chosen for normal points ensure that if a network has no lakes, the integer string is identical with Shreve's binary string for that network.

Properties of Integer Strings

Consider a string of exactly k integers,

$$S = \{s_j \mid j = 1, 2, \cdots, k\}$$

Theorem 1: S represents a valid complete drainage network if and only if

$$M_{1i} > 0 \quad \text{for all} \quad i < k$$

and

$$M_{1k} = 0$$

where

$$M_{pq} = 1 + N_{pq}(0) - N_{pq}(1) + \sum_{m=2}^{s_{\max}} (m - 3)N_{pq}(m)$$

and where $N_{pq}(m)$ denotes the number of elements in the substring $\{s_p, s_{p+1}, \cdots, s_q\}$ for which $s_j = m$ and $s_{\max}$ equals the largest element in S.

Proof: By definition, each element represents a vertex with an out-degree of exactly 1. M_{pq} may thus be seen as enumerating the excess in-degrees (i.e., unused points of attachment), if any, in the substring in question: each fork (0) has an in-degree of exactly 2 and thus increases the excess of in-degrees by 1; each normal source (1) has an in-degree of zero and decreases the count by 1; each lake (m, for some m greater than 1) has an in-degree of $m - 2$ and increases the excess of in-degrees by $m - 3$. The initial term, 1, is included because the out-degree of the first element is not balanced against any in-degree within the substring. Obviously, any string for which $M_{pq} < 0$ cannot represent a valid network. However, if $M_{pq} = 0$, the next element, s_{q+1}, will not be attached to an element in that string but to some fork or lake represented by an element s_w, $w < p$. Thus a complete valid drainage network must have $M_{1k} = 0$ and also $M_{1i} > 0$ for $i < k$.

Notation: $N(m)$, without subscripts, will hereafter be used to denote the number of elements in the entire string S for which $s_j = m$.

Additional Definitions

As will be shown below, the presence of lakes greatly increases the number of possible topologically distinct arrangements. Thus for statistical testing it often becomes necessary to group networks. For channel networks without lakes the aforementioned ambilateral classes have often been used for this purpose. Two more bases for grouping are suggested for networks with lakes.

One is the 'lake-degree set.' For a drainage network containing exactly L lakes, the lake-degree set is defined as the set of exactly L nonnegative integers denoting the in-degrees of the lakes. By convention, these will be listed in ascending order.

Following *Smart and Werner* [1973], a 'path' is the shortest route between a particular vertex in a channel network and the network outlet. We may then define the 'lake-path subgraph' of a network as the set of all paths commencing at lakes, together with the lakes, the outlet vertex, and any junctions at which distinct lake-paths merge. Two networks are said to be members of a lake-path identity set (LPIS) if, and only if, their lake-path subgraphs are topologically identical plane trees. Networks with two lakes and one source arranged by LDS and LPIS are shown in Figure 2.

Some Combinatorial Results

De Bruijn and Morselt [1967] have shown that the number of distinct plane trees with m lines (in stream networks, links) is given by

$$\frac{1}{m}\binom{2m-2}{m-1}$$

This assumes that all vertices of a given degree are identical. For stream networks, and with no restrictions on lake in-degrees, we could simply multiply this by terms representing the number of ways the point sources could be distributed among the vertices of degree 1, and the forks among degree 3 vertices. However, since there are fairly strong geomorphological constraints on lake in-degrees within a region, the following approach may be more useful.

Any string S can be reduced to a related binary string by ignoring the lakes in the network. Any 2 becomes a 1 in this transition. Similarly, any 4 becomes a 0, indicating a simple junction of the lake's two inlets. A 3 is deleted, the inlet and outlet being represented by whatever binary integer follows the 3 in the string. A lake of in-degree $(m - 2)$ greater than 2, represented by an integer m greater than 4, cannot be reduced uniquely; one possible reduction substitutes exactly $m - 3$ 0's.

These rules can be used to obtain the number of topologically distinct networks with a particular lake-degree set and with $s_{max} \leq 4$. Consider a binary string with μ 1's and $\mu - 1$ 0's, where $\mu = N(1) + N(2)$. *Shreve* [1967] gave the corresponding number of topologically distinct trees as

$$P = \frac{1}{2\mu - 1}\binom{2\mu - 1}{\mu}$$

Now consider all those strings with $s_{max} \leq 4$ which would reduce to binary strings with μ 1's if the lakes were replaced, following the transition rules. The $N(2)$ 2's can be substituted for any combination of the μ 1's; thus for each of the P TDCN's the number of distinct networks which may be formed by inserting the 2's is given by

$$\binom{\mu}{N(2)}$$

Similarly, the $N(4)$ 4's may be substituted for any combination of the $\mu - 1$ 0's in

$$\binom{\mu - 1}{N(4)}$$

distinct ways.

Any number of the $N(3)$ 3's may be placed in front of any integer in the string, creating a string of length $2\mu - 1 + N(3)$. However, the final position must be occupied by a 1 or a 2, leaving $2\mu - 2 + N(3)$ feasible locations. The number of

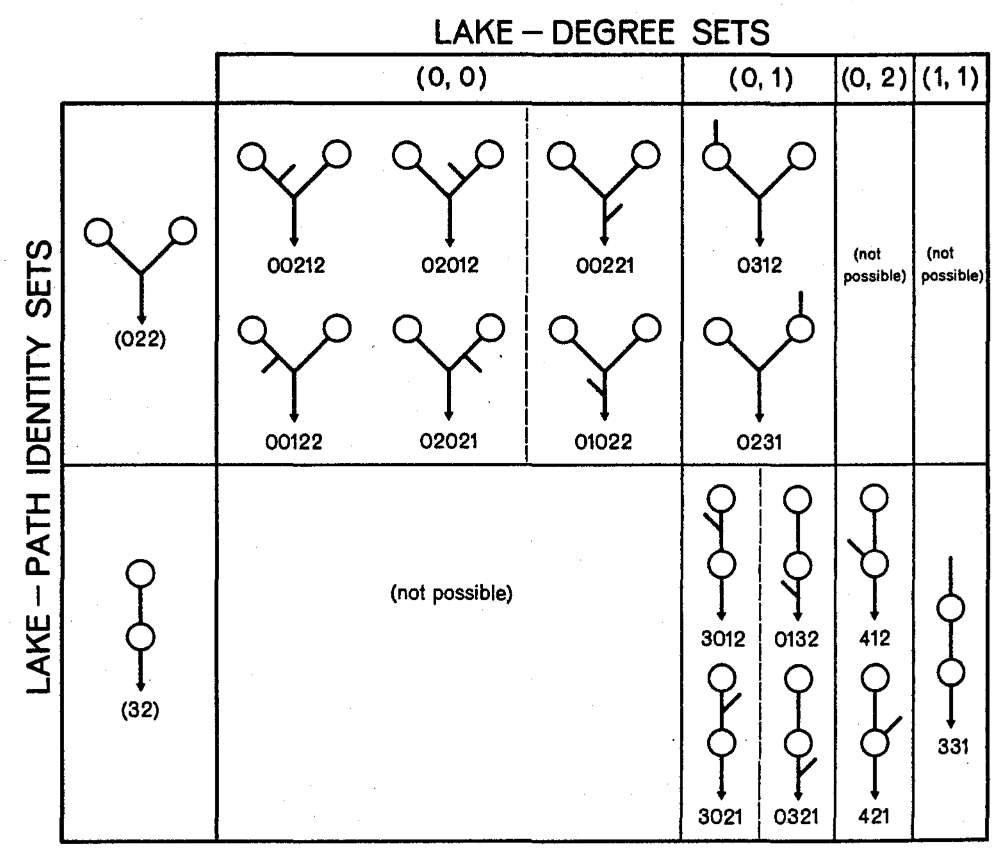

Fig. 2. The 15 possible topologically distinct drainage networks which can be formed from exactly two lakes and one normal source.

TABLE 1. Maximum Possible Number of Distinct Lake-Degree Sets With $L \le 10$ Lakes and $N(1) \le 10$ Normal Sources

	$N(1)$										
L	0	1	2	3	4	5	6	7	8	9	10
0*	1	1	1	1	1	1	1	1	1	1	1
1	1	2	3	4	5	6	7	8	9	10	11
2	2	4	6	9	12	16	20	25	30	36	42
3	4	7	11	16	23	31	41	53	67	83	102
4	7	12	18	27	38	53	71	94	121	155	194
5	12	19	29	42	60	83	113	150	197	254	324
6	19	30	44	64	90	125	169	227	298	388	498
7	30	45	66	94	132	181	246	328	433	566	730
8	45	67	96	136	188	258	347	463	609	795	1025
9	67	97	138	192	265	359	482	639	840	1092	1410
10	97	139	194	269	366	494	658	870	1137	1477	1900

* With zero lakes the lake-degree set is the null or empty set.

possible arrangements can be found by considering the number of distinct ways in which 3's can be placed in $N(3)$ of these locations, the remaining $2\mu - 2$ positions plus the last one being occupied by the original string. The number of ways is thus

$$\binom{2\mu - 2 + N(3)}{N(3)}$$

Note that the implied condition $N(1) + N(2) = N(0) + N(4) + 1$ follows from theorem 1. Thus the number of distinct networks with particular $N(0)$, $N(1)$, $N(2)$, $N(3)$, $N(4)$, and $s_{max} \le 4$ is given by

$$\frac{1}{T}\binom{T}{N(1) + N(2)}\binom{N(1) + N(2)}{N(2)} \cdot \binom{N(0) + N(4)}{N(4)}\binom{T - 1 + N(3)}{N(3)}$$

where

$$T = N(0) + N(1) + N(2) + N(4) = 2\mu - 1$$

The number of topologically distinct networks in a given lake-degree set is obtained by summing over all possible values of $N(1)$ (or $N(0)$) and is clearly infinite.

The number of possible distinct lake-degree sets for a network with L lakes will of course depend on the number of inputs to lakes. This is clearly a problem of partitioning. One of the best known partitioning problems involves dividing an integer n into at most m parts. The number of distinct ways in which this can be done is conventionally denoted as $p(n, m)$ and has been tabulated by *Gupta* [1962]. Dividing n into at most m parts is equivalent to defining a set of exactly m integers, some of which may be zeros, whose sum is exactly n. Thus if the total number of inputs to lakes is exactly I, the number of distinct lake-degree sets is given by $p(I, L)$. There is no closed expression for $p(n, m)$; values can be obtained by a recursive formula first developed by Euler in 1748 [*Gupta*, 1962] and repeated by *Gupta* [1962, p. ix]. The latter reference tabulates $p(n, m)$ for $n \le 200$, $m \le 100$, and also for $201 \le n \le 400$, $m \le 50$.

If only the number of lakes (L) and the number of sources ($N(1)$) are specified, the number of inputs to lakes may range from zero to a maximum of $L + N(1) - 1$, depending on the number of forks. In this case the number of distinct lake-degree sets is given by

$$\Pi[L, N(1)] = \sum_{I=0}^{L+N(1)-1} p(I, L)$$

Values of $\Pi[L, N(1)]$ for $L \le 10$, $N(1) \le 10$ are given in Ta 1. As an example, for networks with exactly three lakes a two normal sources, there are 11 distinct lake-degree se these include a total of 485 topologically distinct networ

Generating 'Topologically Random' Networks

In order to test certain hypotheses about larger network may be necessary to generate either all possible networks a given size or a random sample from this population. T best approach to this at present would seem to be to perm or shuffle the integer string.

Beginning with a valid string, pairs of string elements selected, and interchanged, as long as the exchange does produce an invalid string. If a vector is generated whose element is M_{1J}, then an exchange of two elements c influences values of that vector between the elements, hence the validity check is restricted to that part of S. P of points to be interchanged may be selected so as produce all valid string permutations [*Robson*, 1969] o produce a randomly shuffled string.

TABLE 2. Observed Number of Subnetworks in the Marchir River Basin Having 11 Links or Fewer, by Number of Lakes (L Number of Normal Sources ($N(1)$) (Note That These Are Nc Independent)

	$N(1)$					
L	0	1	2	3	4	5
0	X	182	27	4	3	1
1	105	23	9	8	3	2
2	42	10	5	3	2	3
3	15	2	3	1	1	0
4	13	1	0	2	0	0
5	12	2	0	0	0	0
6	4	1	1	0	0	0
7	6	1	0	0	0	X
8	3	1	0	0	X	X
9	3	1	0	X	X	X
10	4	0	X	X	X	X
11	1	X	X	X	X	X

X means not possible with at most 11 links.

TABLE 3. Observed Link-Type Probabilities for Marchington River

Type	Probability
0	0.242
1	0.305
2	0.176
3	0.182
4	0.058
5	0.015
6 or higher	0.020

AN EMPIRICAL TEST

As a first test of the topological model for networks with lakes, data were obtained on the topology of a natural river basin. Based on a visual inspection of several 1:50,000 topographic maps from northwestern Ontario, an area was selected which had many lakes, no extremely large lake (or parts thereof), and a minimum of overt structural control. The Marchington River above Stranger Lake (grid location 079683, map sheet 52J/6) drains an area of approximately 830 km^2. As interpreted by the cartographer, the connected network consists of 596 links; it includes 270 lakes, 182 normal sources, and 144 forks. The (topologically) largest lake (Kashaweogama Lake) has an in-degree of 18: the basin also contains 31 small lakes with neither inlets nor outlets; as these are not integrated into the network, they are not further considered here.

A computer program was written to extract subnetworks of various sizes or characteristics. Briefly, the program begins at any link not represented by a 1 or a 2 and then scans ahead a certain number of links in the integer string, testing to determine whether or not a complete string has been found. If one is completed, the string is printed, and the resulting subnetwork topologies can be compared. Table 2 lists frequencies of subnetworks by type.

Hypotheses

Shreve [1967, p. 183] states that in a binary string representing an infinite topologically random network the character representing an exterior link (in our notation a 1) 'appears at any specified position with probability 1/2 regardless of the pattern anywhere else in the sequence.' From this property, one can derive the hypothesis that all topologically distinct subnetworks with a given number of sources (a particular magnitude) are equally likely to occur.

Initially, we attempted to generalize the verbal expression of topological randomness to identify an appropriate set of graphs all of which would be expected to be equally likely to occur. None of the attempted generalizations correctly predicted the frequencies of subnetworks having specified numbers of lakes and of normal sources.

The possibility that the Marchington River network departed significantly from topological randomness was further investigated by performing random permutations of the string. These permutations did not notably alter the relative frequencies of subnetworks falling into various lake-degree sets, indicating that the constraints on subnetworks were not geometric.

A generalized version of Shreve's statement concerning infinite topologically random networks is as follows:

> In the integer string representing an infinite topologically random network including lakes, each link-type code appears at any specified position with a constant probability which is independent of position in the string, regardless of the pattern anywhere else in the sequence.

If the individual link-type probabilities are available, this leads to the hypothesis that the probability of any set of subnetworks is equal to the sum of the probabilities of its members, where each member's probability is simply the product of the included link-type probabilities. Note that under this hypothesis, all substrings consisting of the same set of elements are expected to occur with equal frequency;

TABLE 4. Expected and Observed Frequencies of Lake-Degree Sets (LDS), Marchington River

LDS	Calculated Number*	Calculated Percent	Simulated Number†	Simulated Percent	Observed Number	Observed Percent
			One Lake, One Source			
[0]	15.5	32	14.5	28	8	35
[1]	33.3	68	37.5	72	15	65
Total	48.8	100	51.9	100	23	100
			Two Lakes, No Source			
[00]	4.5	19	4.9	21	8	19
[01]	19.2	81	18.3	79	34	81
Total	23.7	100	23.2	100	42	100
			Two Lakes, One Source			
[00]	2.0	10	2.0	9	1	10
[01]	8.5	42	9.1	42	4	40
[02]	3.8	18	5.0	23	1	10
[11]	6.1	30	5.5	25	4	40
Total	20.3	100	21.6	100	10	100
			Three Lakes, No Source			
[000]	0.4	5	0.2	3	0	0
[001]	2.4	33	2.3	29	3	20
[002]	1.1	15	1.2	15	2	13
[011]	3.5	47	4.1	52	10	67
Total	7.4	100	7.8	100	15	100

* Calculated probabilities were multiplied by $k = 596$ to obtain these expected frequencies.
† This is the mean frequency for 10 simulation runs.

within a lake-degree set, all subnetworks with a given number of sources would be equiprobable.

When the observed link-type probabilities (Table 3) were used, the predictions based on this generalization of topological randomness concurred well with the results of the random permutations (see Table 4) and with the relative frequencies observed in the original network. The lake-degree sets in which there are no inputs to lakes are seen to occur infrequently simply because they have more forks and hence more string elements, all of which must have link-type probabilities less than 1. On a local scale within the network, Marchington River's smaller subnetworks do seem to be topologically random.

Why do certain classes (for example, one lake, one source) occur less frequently than predicted, and others (for example, two lakes, no sources) occur more frequently? During the collection of the data it was observed that some subbasins had large numbers of lakes, while others were almost entirely lake-free. If some parts of the network had more lakes than others, this would increase the probability of lake-rich or lake-poor subnetworks, at the same time reducing the chance of mixed categories. The sequence of 'lake' (2, 3, 4, etc.) and 'nonlake' (0, 1) elements in the Marchington River integer string was tested using a 2×2 contingency table; the hypothesis that successive elements were independent with respect to this classification was rejected (chi-square value of 50.46 with one degree of freedom), confirming the impression noted above. The distribution of lake and nonlake elements within the basin is probably controlled by the surficial geology of the region and is a significant nonrandom element in the total network.

SUMMARY AND DISCUSSION

The keystone of *Shreve*'s [1966, 1967, 1975] probabilistic-topologic approach is the postulate that all topologically distinct networks with a given number of sources are equally likely to occur. This postulate can be deduced mathematically from the general hypothesis presented above.

Shreve's original model went on from purely topological properties to include a second postulate regarding link lengths [*Shreve*, 1969], which led to estimates of many aspects of basin geometry (for example, see *Shreve* [1974]). Incorporation of the geometric factor into our model for networks with lakes is more complicated. Not only are there six fundamental link types (normal source, fork, or lake at upstream end; fork or lake at downstream end) rather than two (exterior link, interior link), but also the lakes themselves contribute to stream lengths and other aspects of basin geometry. Work on link lengths and lake geometry is already in progress.

The chief weakness of the probability model presented here is its present reliance on empirical estimates of the frequencies of link types, in particular the probability density function for the lake in-degrees. The number of inlet streams for a particular lake should be closely related to the geometric size of the lake. Thus it should be possible to estimate lake in-degrees from the distribution of lake areas in a region. We thus envisage a more refined model which would estimate frequencies of network topological classes, based only on a 'lake: link ratio' and plus the parameters of the lake in-degree model. These parameters would represent basic constraints on the randomness of drainage networks.

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