SOME LEAST-COST MODELS OF SPATIAL ADMINISTRATIVE SYSTEMS IN SOUTHERN ONTARIO

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ABSTRACT. The paper reformulates the classical Transportation problem to define administrative areas in Southern Ontario. With given centres and stated constraints on the population to be served by each centre, boundaries have been delimited. The authors then develop an iterative method for relocating the centres which reduces the overall cost of the regionalisation system. Three indices for each system have been devised. The regionalisation scheme of the Ontario Hydro Commission has been used for comparative purposes.

Introduction

The agglomeration of phenomena, in order to satisfy agreed objectives, is a fundamental problem. The delimitation of regions, so as to minimise the internal variation among selected parameters, and to maximise the between region variation, is a traditional problem in geography. The problem has been treated quantitatively and and qualitatively (review in Lankford, 1968). The central place theory provides an example where postulates about human behavior are used to derive patterns of human location. Haggett (1965) considers some developments of central place theory in the light of 'field theory', territoriality and packing. Concerned with the size and shape of organisational units, the emphasis, in the studies he examines, is predominantly on the formulation and testing of hypotheses. The hypotheses have been formulated with the aim of helping researchers understand a little more about the processes which appear to operate when man divides space into functional regions.

The problem of spatial division is, in a mathematical sense, extremely complex. Socio-economic prameters are usually discretely distributed over space, and though the mathematical analysis of large discrete systems has improved in the past decade, the methods are still cumbersome. Iterative numerical methods involving large quantities of computational time are typical of this area of study.

This paper is concerned with the examination of the division of space for administrative purposes, and suggests some methods by which objective governmental decisions on questions of cost, individual travel, and division of work load, may be transformed into boundary delineation and the location of administrative centres. Particularly, we are concerned with algorithms for dividing space which attempt to minimise the cost of administering the system, given a set of constraints.

We note that Prescott (1965), following a review of some of the geographical literature concerning administrative areas, concluded that the geographer can make a useful contribution to the planning of internal boundaries (i.e. those within a state) through more exact methods for their analysis. Further, he started that the revision of boundaries, in the light of such detailed research, could result in increased efficiency and economy.

There is another purpose to this study. Actual systems of administrative units, formulated by agencies on an intuitive basis, often with ill-defined objectives, may be compared to the model systems which result from clear, albeit restricted, objectives. For example Whebell (1968), from studies of Ontario and Nova Scotia, has proposed a five-stage theory of politico-territorial evolution whose qualitative concepts are approximately equivalent to the quantitative objectives used in this work.

Underlying this work is the postulate that an administrative service is a good which is produced and consumed according to some general principles. We are attempting to examine some of the spatial elements among the total set of variables which influence the production and consumption of a public service.

Administrative Areas: an example

A simple example will illustrate the relationship between the requirements of the system formulated by the political decision-makers, and the boundary lines drawn on the map. Let us suppose that a number of cities have been designated as administrative centres, and a population which requires a service that is provided by a centre is distributed around these cities. Further, that it is found desirable to draw the boundaries between administrative areas in such a way that the aggregate distance between individuals and their assigned administrative centre is minimised. There are no constraints placed on the number of persons served by each administrative centre.

To achieve the declared objective, clearly individuals are assigned to their nearest centre. Geometrically, the administrative areas appear as irregular polygons whose segments are the perpendicular bisectors of lines joining pairs of centres. In the general case all polygon corners are also junction of three separate areas. Let us call this first allocation model of individuals to their nearest centre the Proximal Solution.

Models of Administrative Systems for Southern Ontario

For this research, Southern Ontario is defined as that part of the province to the south of the French River and Lake Nipissing. In this area, the township is the spatial unit that it used by all government agencies as the fundamental building block for the larger spatial administrative structures. In Southern Ontario there are 590 townships of which 504 are occupied. The population distribution used is punctiform, the inhabitants of each township being assumed to be located at the geometrical centre of the township. We suggest that there are two factors which minimise the effect of this approximation. First, the optimum boundaries run across townships, yet, in any viable system, real boundaries must follow township lines. The divergence from optimality resulting from relocation along township lines outweighs the divergence due to the approximation of the population distribution. Secondly, in the majority of algorithms used in allocation, decisions are made on an ordinal scale of distance. The probability that a small error in population location would result in a boundary revision is therefore low.

Locations of the centres of townships were recorded as two three-figure cartesian coordinates. An analog-digital converter was used to record the locations. The accuracy in location is half a mile on the east-west axis, and one quarter mile on the north-south axis.

It is postulated that the cost of an administrative area is a function of the number of people served and the distance between the consumer of the administrative service and the administrative centre, assuming the centre is the supplier of the service. Explicitly, we hypothesize that the cost C_{ij} of administering a township i from a centre j is a function of the distance d_{ij}, and the population of the township $M_i \cdot A$ simple linear functional relationship is suggested.

$$C_{ij} = M_i \, d_{ij} \tag{1}$$

This relationship is open to criticism. Studies by Chernick and Schneider (1967) on the cost of providing hospital services in relation to distance suggest exponents of 1 or 2 for d_{ij}. Most empirical work on this parameter suggests that d_{ij} can take exponents from one half to three, though other studies favour logarithmic or exponential functions of distance. Malm et al (1966) and Mackay (1960) have discussed functions of M and d in a gravity-model context.

Airline distance, computed from cartesian coordinates, has been used throughout this study. Road distance may be more valid; however, the methods used are not restricted to the relationship expressed in equation 1. Any functional relationship may be used in these models, and no restrictions are placed on the determination of d_{ij}. All distances were computed overland; however, routes which would have crossed major embayments, such as western Lake Ontario, were redefined on a longer path confined to the land area.

The use of a population distribution as database is also arbitrary. Certain agencies may be more concerned with size of area, length of highways, income, education or health attributes of the population than numbers of people per se. These variables could validly be dealt with by the methods employed in the study.

When constraints are placed on the load to be handled by each administrative centre, the problem of administrative division becomes technically more complex. Suppose that in addition to the requirements outlined for the Proximal Solution, each of the centres is required to handle an equal number of people. A recent study by Mills (1967) and earlier studies by a group at the Yale Law School (Weaver and Hess, 1963) consider the delineation of political constituencies (see also Bunge, 1966, and Silva, 1965). In the Weaver and Hess study cost was defined as $M_i d^2_{ij}$, referred to as the Moment of Inertia when

PROXIMAL REGIONS

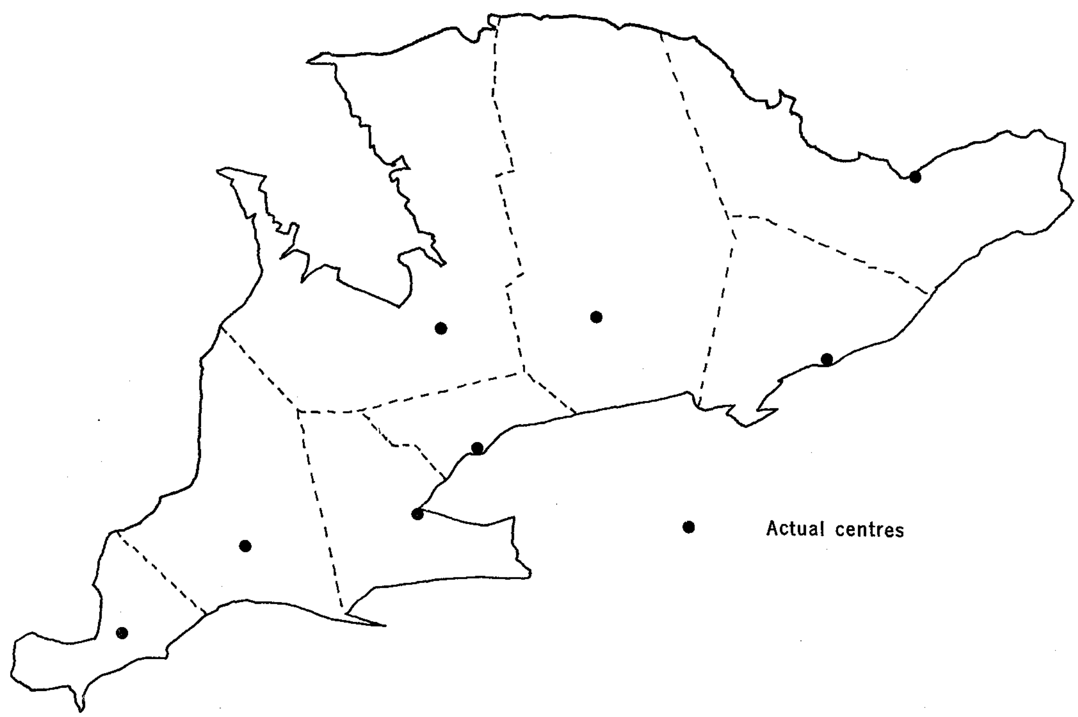

Figure 1. Proximal Solution for the eight centres

summed over the entire population. The mathematical advantages of using the second power of distance will be seen later. Yeates (1963) considered a similar problem, using an algorithm to allocate pupils to schools.

Initial solutions were prepared by the Vogel approximation method, and iterated to an exact optimum by the MODI technique (see for example Reinfeld and Vogel, 1958). Output was prepared on an incremental plotter.

On the basis of the number of centres used for administrative purposes by eight human-service agencies in the province of Ontario, eight centres were selected for this study. The mean of the eight agencies is 8.3 centres. The centres selected were those most frequently used, though Belleville was excluded from the list because of its physical proximity to Kingston. The centres used were London, Ottawa, Hamilton, Kingston, Toronto, Chatham, Peterborough and Barrie.

The proximal regions for these centres are illustrated in Figure 1. This solution is the overall least-cost solution for these centres; any imposition of constraints upon the volume of service handled by the centres will necessarily increase the overall cost, compelling some citizens to travel to more distant centres, as frequently occurs in real systems. By adding the severe constraint that each centre serve an equal number of people, the regions are modified to those shown in Figure 2, the solution of the Transportation problem for the 8 by 504 matrix.

The requirement that an equal population be served by each centre appears unrealistic in the context of Southern Ontario. On the basis of a study of the population served by the eight centres mentioned above (Massam, 1967), the following scaling of administrative work load was introduced into the model.

The Transportation problem was solved again

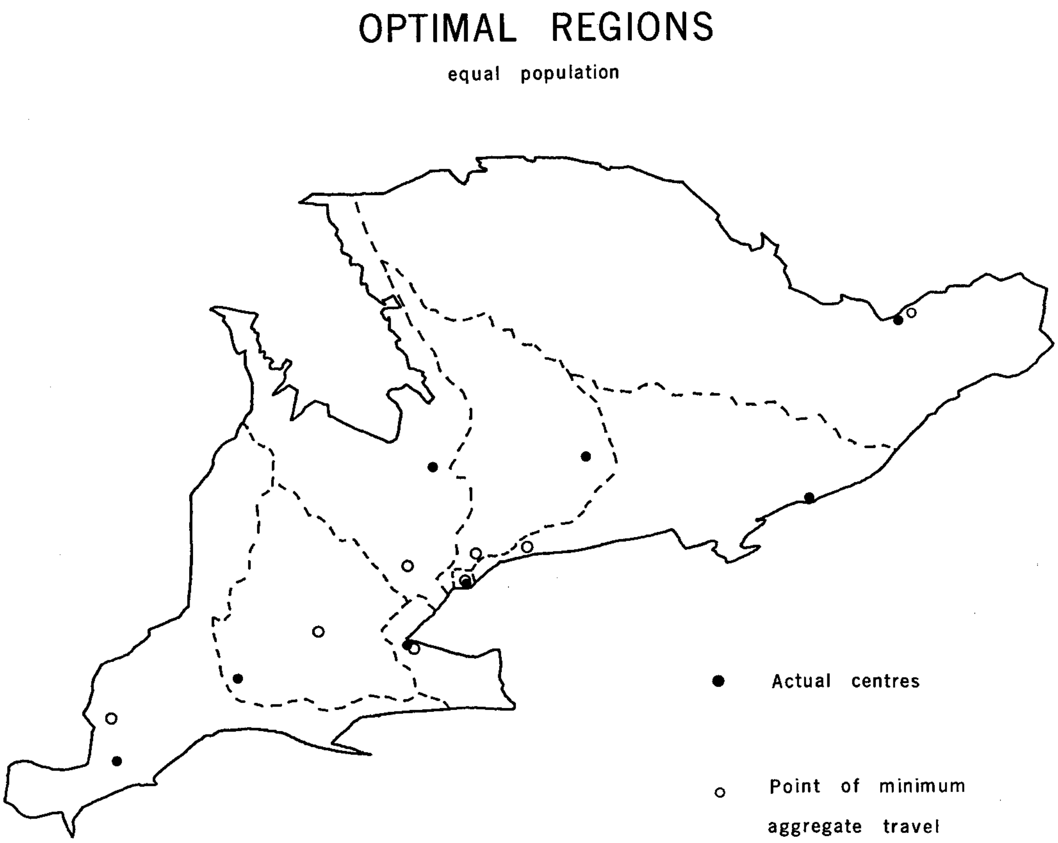

Figure 2. Least cost solution for equal population constraints

Centre	% population served
Toronto	50
London	15
Ottawa	12
Hamilton	12
Chatham	5
Kingston	2
Peterborough	2
Barrie	2

using these constraints. Figure 3 illustrates the optimal solution.

Evaluation of the Models

Three indices were used in the evaluation of these and subsequent models, all based on an average over the entire population. The first index is provided by calculation of the proportion of all persons who are served by the nearest administrative centre, while a second index is obtained from the average distance separating an individual from his administrative centre. Thirdly, the index of efficiency for an area and its associated centre measures the extent to which the position of the centre deviates from the defined optimum point within the area. In this work the optimum location is at the point of minimum aggregate travel for the area, as this point offers the lowest administrative cost by definition. For cost defined as $M_i d^2_{ij}$, the optimum location is at the 'centre of gravity' of the distribution. This point can be readily calculated and an exact solution obtained. With cost defined on a linear basis, however, the point of minimum aggregate travel must be located by a lengthy iterative procedure. Kuhn and Kuenne (1962) devised an algorithm for the solution of this problem which has been applied by Chernick and Schneider (1967) in their analysis of hospital locations. Gould and Leinbach

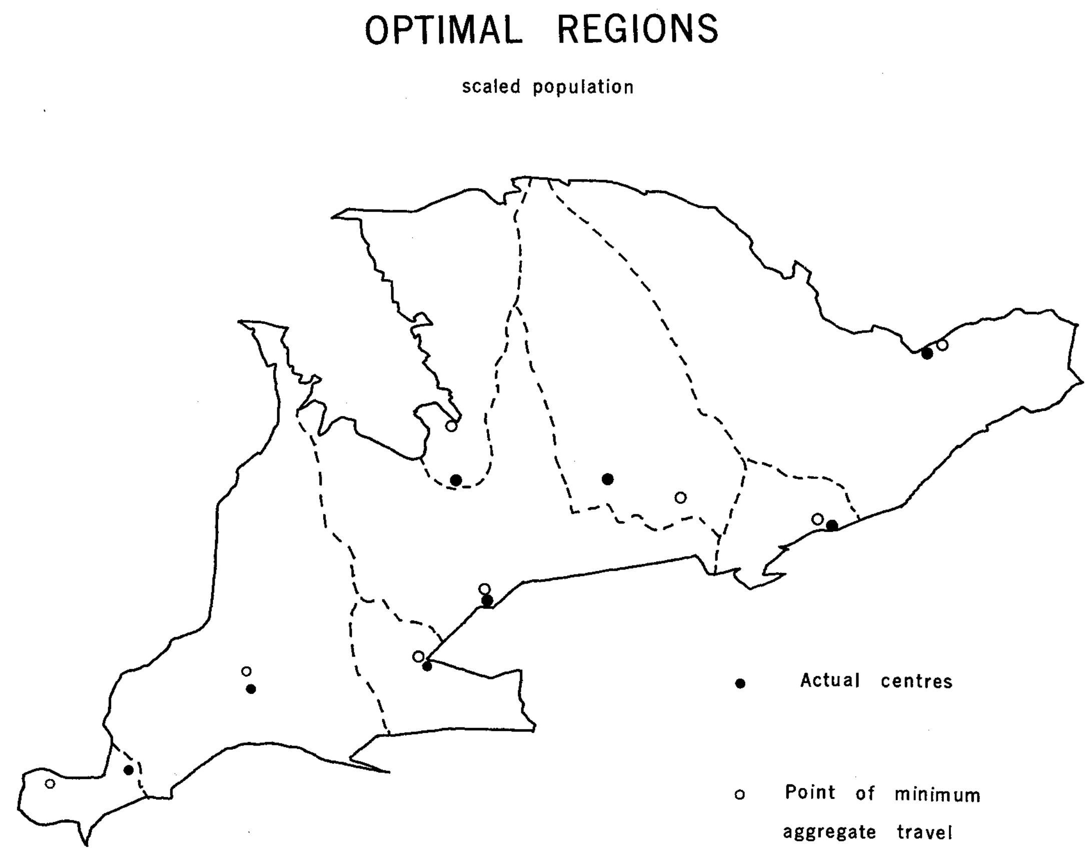

Figure 3. Least cost solution for scaled population constraints

1966) have also used the method to analyse hospital locations in Guatemala. In this work, a more recent, more general method due to Cooper (1967) was used.

The index of efficiency I_j of an area j is defined by the ratio of first moments about the ideal and the actual locations of the centre, i.e.

$$I_j = \frac{\sum_i M_i z_{ij}}{\sum_i M_i d_{ij}}$$

where z_{ij} is used to denote the distance between the township location and the point of minimum aggregate travel. This index varies between 0.0, when the centre is located at infinity, and 1.0 when the centre is located at the optimum point.

The results are shown in the first table, together with the figures from actual administrative systems for comparison. Clearly the proximal and equal population solutions represent opposite extremes in a payoff between cost minimisation and control of the volume of service handled by a centre. The scaled population model, Figure 3, based on realistic constraints derived from actual systems, represents an attempt to compromise these two objectives on the part of the service agencies.

The efficiency indices for the real and model systems indicate that a reduction in cost may be achieved by relocation of the administrative centres closer to the optimum points, thus partially obviating the necessity to compromise objectives. On the figures the optimum locations for administrative centres are shown by open circles. Accordingly, Figure 4 is the result of taking new centres at the optimum points and modifying the boundaries by resolving the Transportation problem under the original constraints. Successive applications of this procedure result in an efficiency index very close to 1.0. The index is unable to

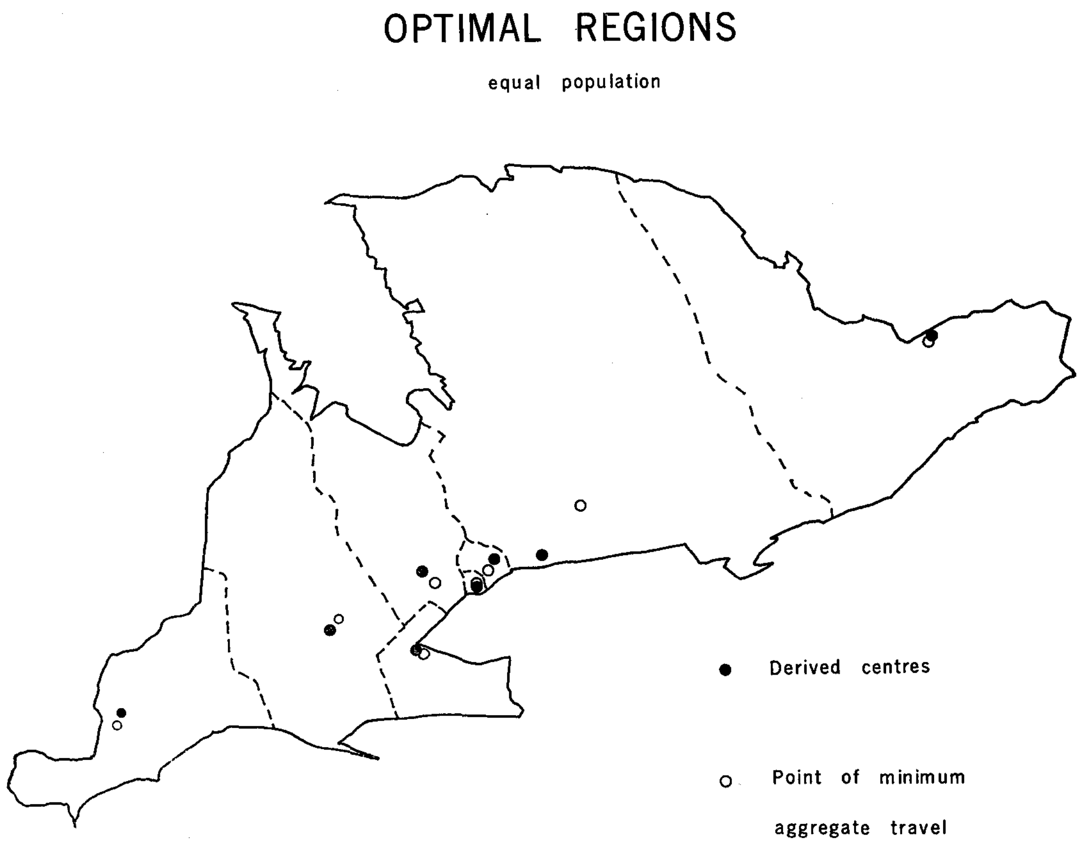

Figure 4. First iteration of centre locations, equal population constraints

Table 1: Evaluation of the Eight-Centre Regionalization Schemes.

Regionalization Scheme	Mean Index of Efficiency	% Served by Nearest Centre	Mean Distance of an individual from his administrative centre in miles
1	0.686	59.8	39.76
2	0.897	77.6	22.10
3	0.961	81.6	20.75
4	0.919	78.1	22.66
5	0.989	85.4	20.64
6	0.997	87.1	19.82
7	0.916	100.0	19.22
8	0.650	54.6	52.51
9	0.555	49.2	39.68

Key for Table 1

1 Least cost solution with equal population constraints
2 First iteration of centre locations, equal populations
3 Final iteration of centre locations, equal populations
4 Least cost solution with scaled populations
5 First iterations of centre locations, scaled populations
6 Final iteration of centre locations, scaled populations
7 Proximal solution for the eight centres
8 Actual scheme used by Ontario Hydro Commission
9 Actual scheme used by Department of Education

reach 1.0 exactly because of the requirement that boundaries be drawn along township lines. Figures 5 and 6 represent the final iterations under the equal and scaled population-sharing constraints. The evaluations of these models are shown in Table 1.

The significance of these results demands investigation. The procedure that has been applied has attempted to minimise total cost by allowing centres and boundaries to be successively relocated. Intuitively, the optimal system is one in which all persons are assigned to their nearest centre, yet examination of Table 1 shows that this state has not been achieved. Two explanations are suggested. First, in a general popula-

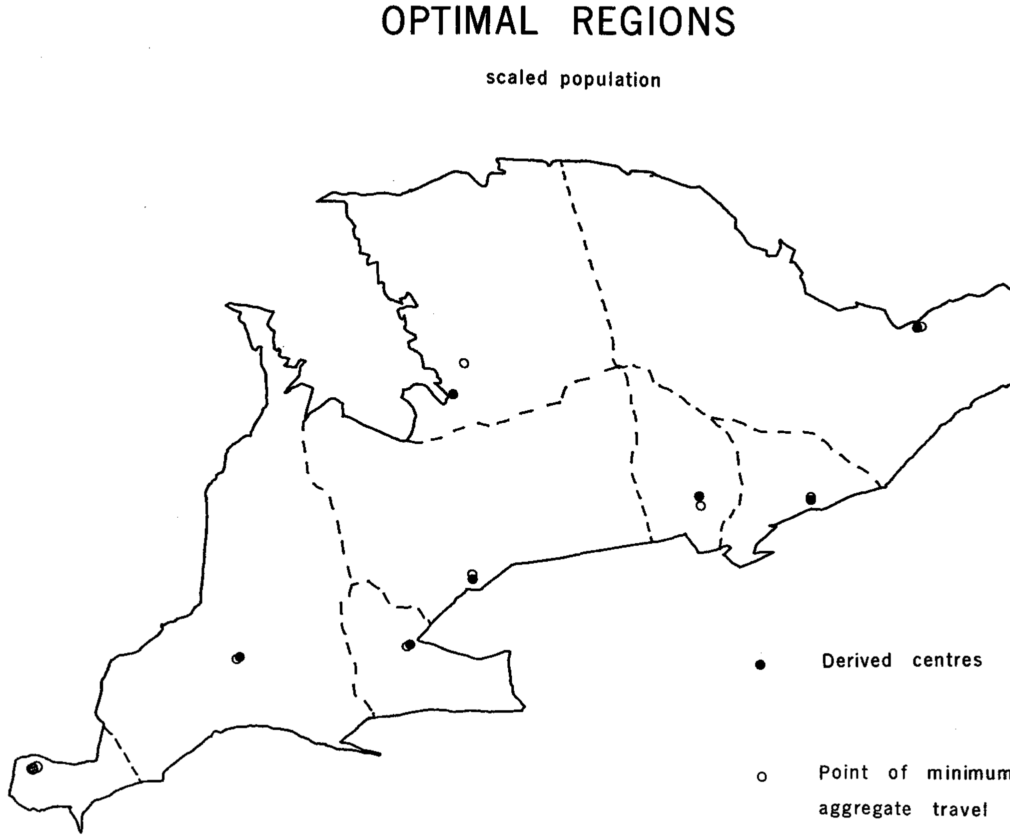

Figure 5. Final iteration of centre locations, scaledconstraints

tion distribution such as that of Southern Ontario, with irregular outline and non-uniform density, such an ideal is not possible. This is readily shown in the case of a one-dimensional population distribution along a line. Second, that the procedure is not capable of finding the overall, global minimum cost solution.

The objective function of the procedure may be written thus. Find A_{ij}, X_j and Y_j such that the function

$$\sum_i \sum_j A_{ij}\left[(x_i-X_j)^2 + (y_i-Y_j)^2\right]^{1/2}$$

is minimised. A_{ij} represents the population assigned from township i to centre j. X_j and Y_j are the coordinates of centre j; x_i and y_i are the coordinates of township i. The constraints applied are

$\sum_i A_{ij}=M_i$ and $\sum_j A_{ij}=P_j$, the population to be assigned to the centre j.

In the general case where X_j and Y_j are variable, the problem examined above, the objective function is non-linear. Cooper (1967b) has shown that the objective function is neither convex nor concave; the function possesses local minima as well as the global minimum. Thus the end-point of the iterative procedure depends on the locations of the original centres. Further, there are no adequate procedures capable of solving such problems even approximately, unless they be of trivial size.

It is nevertheless evident from Table 1 that these models, by allowing centres to be relocated, provide a lower total cost than the fixed centre solutions. In Table 2, results are shown from a study of the centres actually used by the Hydro-Electric Power Commission of Ontario. The results of a randomly generated allocation pattern are also shown for comparison. The scaled populations are those at present served by the centres.

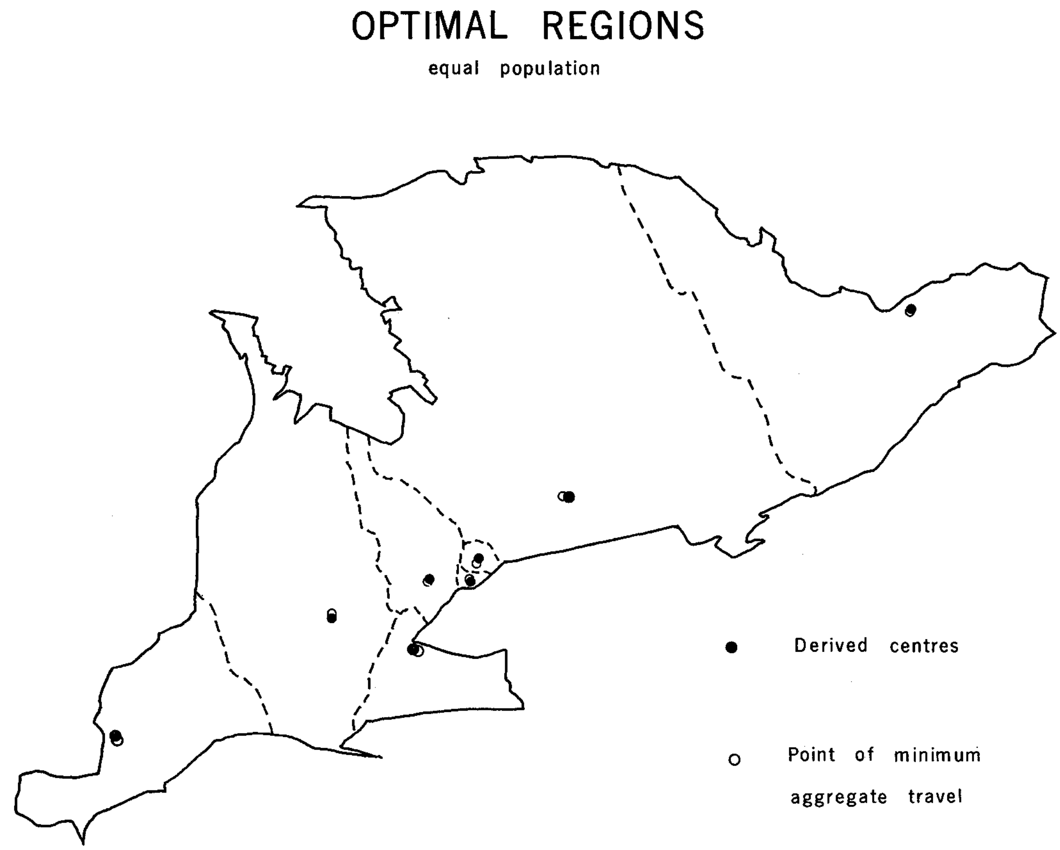

Figure 6. Final iteration of centre locations, equal constraints

Table 2: Hydro Electric Power Commission of Ontario.
Distance between an individual and his administrative centre in miles.

Centre	Regionalization Schemes								
	1	2	3	4	5	6	7	8	9
Barrie	63.7	42.0	42.2	44.4	37.4	36.7	37.1	87.9	81.9
Belleville	79.3	78.8	80.2	81.3	52.9	53.2	53.2	139.7	121.4
Hamilton	23.4	21.0	21.6	21.2	19.6	20.2	19.2	71.4	70.3
London	48.2	47.1	47.3	47.3	41.8	41.8	41.7	159.1	106.9
Toronto	48.2	8.9	3.3	10.9	10.9	10.4	11.1	41.2	75.0
Weighted Mean	49.7	32.6	38.1	33.2	26.6	26.5	26.5	98.1	88.8
% Population served by nearest centre	54.6	100.0	79.0	94.7	90.2	94.5	90.2	19.6	22.4

Key for Table 2

1 System currently used by the Commission
2 Proximal solution
3 Least cost solution with equal population constraints
4 Least cost solution with scaled population constraints
5 Scaled populations: first iteration of centre locations
6 Second iteration of centre locations
7 Final iteration of centre locations
8 Random allocation of townships to centres, equal population constraint
9 Random allocation, scaled population constraint

The techniques make no assumptions about the nature of the defined data-base, whether population or any other parameter of the area units, nor about the shape of its density surface.

The problem examined here is a general case of the well-known Weber location problem. When no constraints are placed upon the population administered by each centre, the problem of locating centres to achieve the minimum cost becomes much simpler, and has been treated by many workers (for review see Scott, 1968). Algorithms have been developed, the most successful of which are capable of handling systems of considerable size with reasonable computation times and satisfactory accuracy. The problem has also been tackled by biologists as a special case of taxonomic description, in which the distances between points are taken as the characteristic parameter.

References

Bunge, W., 1966: "Gerrymandering, Geography, and Grouping", *Geographical Review*, Vol. 56, No. 2, pp. 256–263.

Chernick, H. D., and J. B. Schneider, 1967: "A New Approach to the Delineation of Hospital Service Regions", Discussion Paper Series No. 16, Regional Science Research Institute, Philadelphia.

Cooper, L., 1967: "An Extension of the Generalized Weber Problem", Publication of the Department of Applied Mathematics, Washington University, St. Louis.

Cooper, L., 1967b: "Solutions of Generalized Locational Equilibrium Models", *Journal of Regional Science*, Vol. 7, No. 1, pp. 1–8.

Gould, P. R., and Leinbach, T. R., 1966: "An Approach to the Geographical Assignment of Hospital Services", *Tijdschrift voor Econ. En Soc. Geografie*, Sept/Oct. Vol. 57, pp. 203–206.

Haggett, P., 1965: "*Locational Analysis in Human Geography*", London, pp. 40 ff.

Kuhn, H. W., and Kuenne, R. E., 1962: "An Efficient Algorithm for the Numerical Solution of the Generalized Weber Problem in Spatial Economics", *Journal of Regional Science*, Vol. 4, No. 2, pp. 21–33.

Lankford, P. M., 1968: "Regionalization: Theory and Alternative Algorithms", M. A. Dissertation, Department of Geography, University of Chicago.

Mackay, J. R., 1960: "The Interactance Hypothesis and Boundaries in Canada: A Preliminary Study", *The Canadian Geographer*, Vol. 4, No. 1, pp. 1–8.

Malm, R., et al, 1966: "Approaches to Simulations of Urban Growth", *Geografiska Annaler*, Vol. 48B, No. 1, pp. 9–22.

Massam, B. H., 1967: "An Analysis of Administrative Areas in Southern Ontario", M. A. Dissertation, Department of Geography, McMaster University.

Mills, G., 1967: "The Determination of Local Government Electoral Boundaries", *Operational Research Quarterly*, Vol. 18, No. 3, pp. 243–255.

Prescott, J. R. V., 1965: "*The Geography of Frontiers and Boundaries*", London, pp. 152–182.

Reinfeld, N. V., and Vogel, W. R., 1958: "Mathematical *Programming*", New Jersey, pp. 180ff.

Scott, A. J., 1968: "On the Optimal Partitioning of Spatially Distributed Point Sets", Publication of the Department of Town Planning, University College, London.

Silva, R. C., 1965: "Reapportionment and Redistricting", *Scientific American*, Vol. 213, No. 5, pp. 20–27.

Weaver, J. B., and Hess, S. W., 1963: "A Procedure for Non-Partisan Districting: Development of Computer Techniques", *Yale Law Journal*, Vol. 73, pp. 288–308.

Whebell, C. F. J., 1968: "Core Areas in Intra-State Political Organization", *Canadian Geographer*, Vol. 12, No. 2, pp. 99–112.

Yeates, M., 1963: "Hinterland Delimitation–A Distance Minimising Approach", *The Professional Geographer*, Vol. 15, No. 6, pp. 7–10.