Michael F. Goodchild

The Aggregation Problem in Location-Allocation

Location-allocation solutions based on aggregate estimates of demand are subject to error because of a loss of locational information during aggregation. It is shown that any method to remove or reduce uncertainty must be solution-specific and therefore impractical, for both median and center classes of problems. The significance of the error is illustrated by simulation of solutions to a number of artificial and real problems. It is suggested that aggregation problems be specifically addressed in applications of location-allocation models, and possible methods are proposed.

1. INTRODUCTION

Uncertainty can be present in location-allocation models in a number of ways. Lack of information, measurement error, and uncertain forecasts of future patterns can affect the data base on which the model rests, the constraints, and even the objective function. All of these, in turn, result in alternative solutions with varying degrees of likelihood. Most of the existing work on uncertainty appears to have concentrated on the data base; although Webber [*28*] has provided a review of some of the wider effects of uncertainty. A number of papers (see for example [*6*,*14*,*30*]) have studied the effects of uncertainty in the knowledge of the precise location of each demand node, deriving expected solution locations under various distribution assumptions. Carbone [*3*] has applied stochastic programming techniques to the case of uncertain demands, and Jucker and Carlson [*13*] have considered this class of uncertainty in the plant location problem. Frank [*7*,*8*] developed some theoretical results for probabilistic weights on networks, and Wesolowsky [*29*] provides a more or less complete solution to the one-dimensional facility location problem under uncertainty. Seppala [*23*] has applied stochastic programming to the case of uncertain costs.

Many of the fields to which location-allocation models have been applied take the data base as some aggregation of a geographically dispersed demand.

Michael F. Goodchild is professor of geography, University of Western Ontario

0016-7363/79/0779-0240$00.50/0
GEOGRAPHICAL ANALYSIS, vol. 11, no. 3 (July 1979)
Submitted 8/78. Revised version accepted 10/78.

When the demand is based on a highly dispersed pattern of individuals, as it is in problems of public facility location, it is necessary in the interests of manageability to aggregate to some set of more or less arbitrary statistical areas. The same is true of emergency facility problems, where a virtually continuous probability surface of demand must be collapsed onto a number of zones. The zones themselves may be chosen to coincide with standard statistical areas, such as census tracts, to allow predictive models to be built, or may be chosen to allow easy administrative implementation of the plan through the use of administrative "building blocks."

A growing literature has begun to show that many results of geographical modeling are not invariant under changes in the system of geographical areas. It is possible to identify fields, in fact, where results are almost entirely attributable to zoning. It has been shown to be particularly important in spatial interaction modeling (see, for example, [*1*, *21*, *19*]) where model errors drop consistently with successive aggregation.

Openshaw [*20*] has argued that the effects of zoning are conventionally treated rather like those of sampling; the observed zones are taken to be a random sample of all possible sets of zones, and hence the results are taken to be a sample of all possible sets of results. This is unacceptable, because zones are always constructed for some specific purpose that is rarely irrelevant to the study, and so usually have a strong biasing effect. It is therefore imperative that the effects of zoning on such measures as ecological correlations be examined explicitly, and he gives examples of possible approaches.

Tornqvist et al. [*25*] have discussed the aggregation problem in a location-allocation context. This paper attempts to evaluate the nature and degree of zoning uncertainty in location-allocation solutions, and the ways in which it can be minimized. The first section discusses aggregation effects on median problems in continuous space. This is followed by a comparison with minimax and coverage solutions, and then by a discussion of discrete space results.

2. AGGREGATION EFFECTS IN CONTINUOUS SPACE: MEDIANS

Let there be a demand base consisting of n points located at $(x_i, y_i, i=1,n)$ with weights w_i.

Define a median as the point (x_m, y_m) such that

$$\sum_i w_i\left[(x_i - x_m)^2 + (y_i - y_m)^2\right]^{1/2}$$

is minimized. At this median, the first derivative conditions reduce to

$$\sum_i w_i \cos\theta_i = 0 \qquad \sum_i w_i \sin\theta_i = 0,$$

where θ_i is the angle subtended at the median by the point i and some fixed direction.

Now let a subset S of nodes be aggregated. If the median is to be unaffected, we must have

$$\sum_{i \in S} w_i \cos\theta_i = W\cos\phi; \qquad \sum_{i \in S} w_i \sin\theta_i = W\sin\phi,$$

where $W = \sum_{i \in S} w_i$, and ϕ is the angle subtended by the aggregate node. Since both conditions must be independently satisfied, it is clear that a solution for ϕ will not generally exist. Furthermore, any solution for ϕ depends on the θ_i, which in turn depend on the location of the median. We conclude that there can be no general rule for the location of the aggregate point that depends only on the locations of points in S. As a corollary, replacement of S by an aggregate point will generally affect median solutions. (Equally, if a "safe" aggregation procedure did exist, it would form the basis of a very much simpler solution for the median.)

In aggregating geographical data, the point most commonly chosen is the centroid, $X = \sum x_i w_i / W$; $Y = \sum y_i w_i / W$. Figure 1 shows an aggregation of two points A and B, weight $\propto W$ and $(1-\propto)w$ respectively. The centroid G divides the line AB in the ratio $AG{:}BG = (1-\propto)/\propto$. To preserve the median, however, the two points would have to be replaced by one of weight $W^1 = W[\propto^2 + (1-\propto)^2]^{1/2}$ at an angle λ given by $\tan\lambda = \tan\theta(2\propto - 1)$. We can represent the distortion in vector notation. Let the two nodes A and B generate vectors at M proportional to their respective weights. Then λ and W^1 are respectively the direction and magnitude of the resultant vector. The distortion of the median depends on the difference between the vectors generated by A and B, and that generated by a single aggregate point located at the centroid. Writing X_R and Y_R for the components of the resultant difference, it is easy to

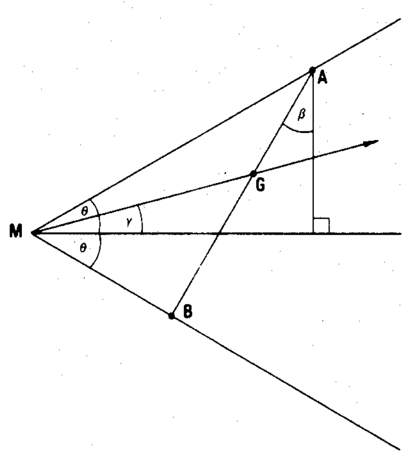

FIG. 1. Aggregation of Two Points in Relation to a Median

show that

$$X_R = W(\cos\theta - \cos\gamma)$$

$$Y_R = W[(2\propto - 1)\sin\theta - \sin\gamma],$$

where

$$\tan\gamma = \tan\theta\left[\frac{(2\propto - 1) + \tan\theta\tan\beta}{1 + (2\propto - 1)\tan\theta\tan\beta}\right]. \qquad (1)$$

The magnitude of the resultant is given by $(X_R{}^2 + Y_R{}^2)^{1/2}$ and its direction by $\tan^{-1}(Y_R/X_R)$.

The most important conclusion to be drawn from equations (1) is that the resultant is never zero except in the trivial case $\propto = 0$ or 1. When $\propto = 0.5$, the centroid G is in the correct direction from the median but for the resultant to vanish G would have to have a reduced weight.

The largest resultants occur when β approaches $-\theta$ and $\propto$ approaches 1. In this region R can exceed W, up to a limit of $2^{1/2}W$ when the centroid and the resultant of A and B are 2θ apart. This creates the paradoxical situation that the distortion vector due to the aggregation of A and B is greater than the sum of the vectors generated by A and B themselves.

Hillsman and Rhoda [*12*] calculated the error that results when distances are measured from some external reference point to a centroid. Distance to the centroid is a systematic underestimate of the mean distance to disaggregated points, and Hillsman and Rhoda showed that the error can be as much as 8 percent. Distance error does not affect median locations directly, however; the source of median error lies in the inappropriate representation of direction and weight in aggregated data.

The next section illustrates the magnitude of the aggregation effect by simulation of a number of median problems in continuous space.

3. SIMULATIONS OF MEDIAN PROBLEMS IN CONTINUOUS SPACE

The disaggregated data base for these experiments is defined as an area 30 units by 30, containing 900 points of unit weight in a square array. Solutions were found for 1, 2, and 4 medians to minimize

$$\sum_i \sum_j I_{ij}\left[(x_i - X_j)^2 + (y_i - Y_j)^2\right]^{1/2} \quad \text{subject to} \quad \sum_j I_{ij} = w_i,$$

where X_j, Y_j are the median locations, and I_{ij} is the allocation from i to j. This is, of course, the simple unconstrained location-allocation problem in continuous space. Solutions were found using the patterned search approach described by Tornqvist et al. [*25*] as programmed by Kohler [*15*] after rejecting the alternating heuristic methods [*5*] because of poor performance. Throughout the study the reliability of heuristics was checked repeatedly by multiple runs to

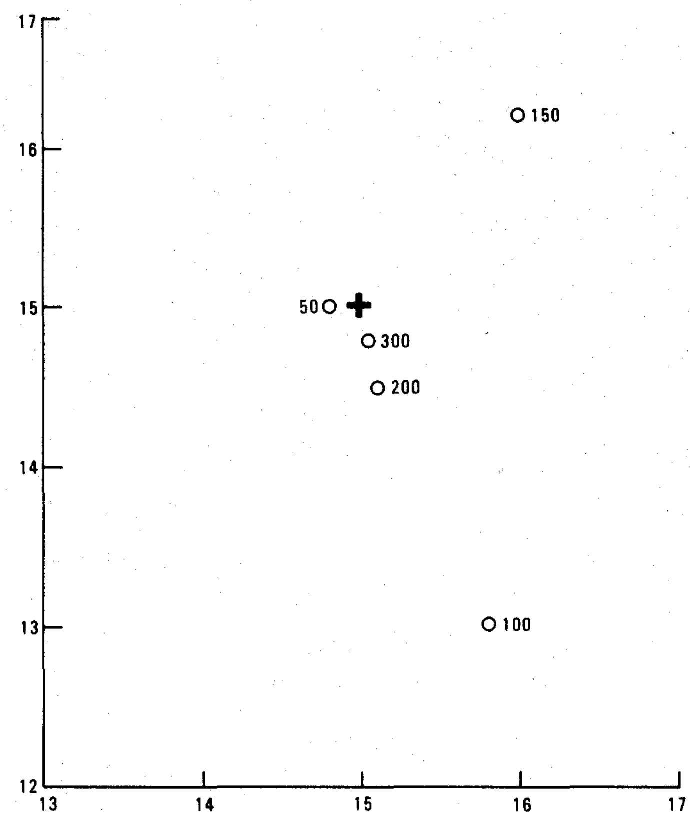

FIG. 2. 1-Median Solutions under Successive Hierarchical Aggregations

ensure that distortions were not likely to be the result of local minimum problems.

The difficulty of defining a random aggregation in space, or a random partitioning, has been recognized for some time. Two approaches were used here: a random hierarchical procedure with contiguity constraints, and the S-mosaic defined by Pielou [*22*]. In the first, which is analogous to the well-known Ward clustering algorithm [*26*], a list was made of all possible ways of joining neighbors, using the Rook's case in which each point has four neighbors. At each stage in the aggregation a join was randomly selected from this list. As groups form, there may be several joins that would result in the aggregation of the same two groups. The algorithm rejects any joins that might result in the linking of two arms of a single group. Under these conditions, n areas will be produced after precisely 900-n joins.

In the S-mosaic process, a set of points are randomly located in the study area. Each base data unit is then assigned to the nearest random point, and becomes part of the latter's aggregate area. The aggregate areas are consequently the Thiessen, Dirichlet, or Voronoi regions around each random point. Since a random point need not be assigned any data points, the number of aggregate areas may be less than the number of random points.

In both modes, aggregations were made to the centroid, since this is the common geographical practice and since it has been shown above that no effective alternative exists.

Figures 2–4 show the results of successive hierarchical aggregations on the 1-, 2-, and 4-median solutions. (In each figure the numbers denote the number of areal units in the aggregated data base.) For 1 median, (15, 15) is the solution for both extremes at $n = 900$ (no aggregation) and $n = 1$ (complete). As aggregation proceeds, the optimum is observed to drift away from (15, 15) and then return. Differences are first observed at around $n = 400$, and reach a maximum at about $n = 100$. Judging by the literature, $n = 100$ would be considered a high degree of disaggregation for a simple, uniform study area.

In the 2- and 4-median solutions the two extremes are not the same. At $n = 900$ each median is assigned a symmetrical share of the study area, but as n tends to 1 one center becomes dominant at (15, 15) and the other(s) disperse to peripheral locations. For the 2-median solution Figure 3 illustrates the very rapid movement of the second, peripheral node as n is reduced below 100.

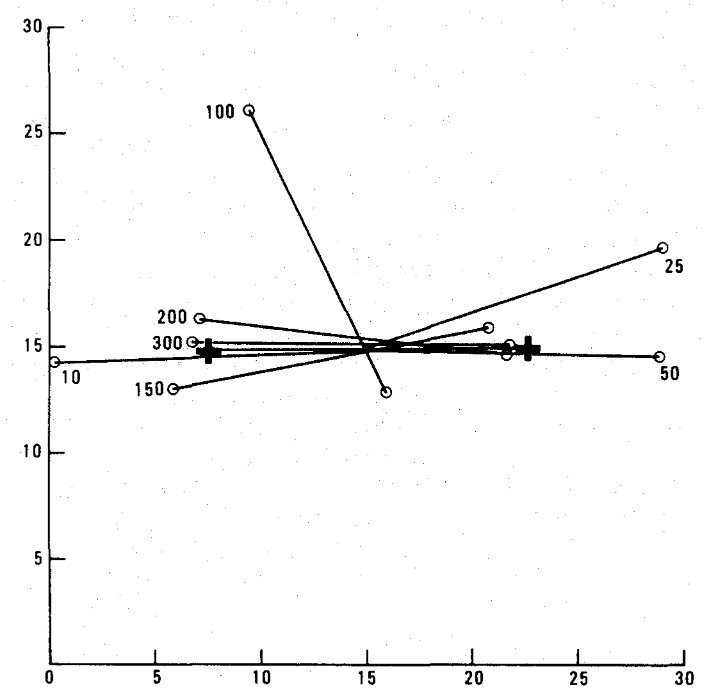

Fig. 3. 2-Median Solutions under Successive Hierarchical Aggregations

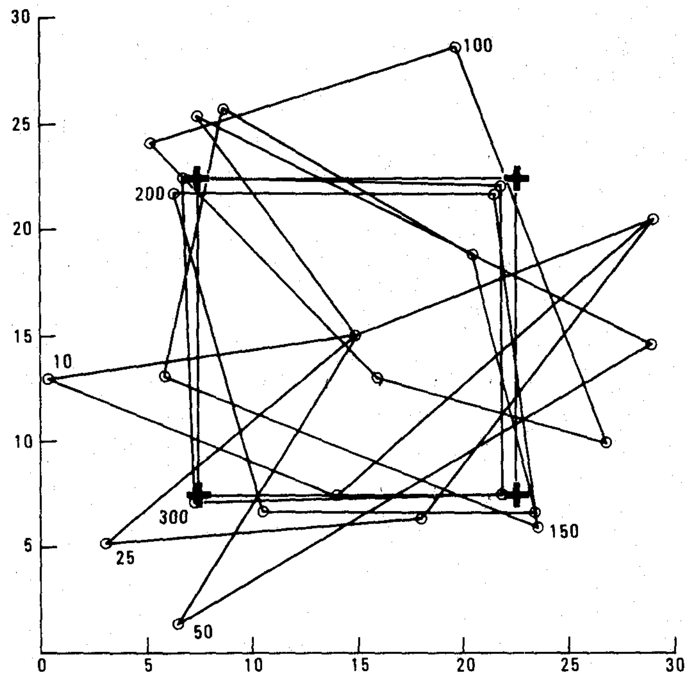

FIG. 4. 4-Median Solutions under Successive Hierarchical Aggregations

Similarly, Figure 4 shows little movement up to $n \sim 150$, followed by a migration of one node to the dominant location and rapid oscillation of the other three.

In contrast to the successive hierarchical aggregations of Figures 2–4, Figure 5 shows a number of independent replications of aggregation to $n = 150$ for the 4-median solutions. Again it is possible to produce large distortions of the solution by manipulating the pattern of geographic zones even at this relatively low level of aggregation.

Of course, the parameters of the preceding simulations have been deliberately chosen to produce large distortions. Effects would be less dramatic with a nonuniform demand density and a radially less symmetric study area, if measured solely in terms of location. But, on the other hand, such problems are much more readily solved by intuition, and much less sensitive to changes in parameters, than the ones used in the simulation. So if distortions are measured in terms of sensitivity, or the performance of intuition, they remain roughly constant.

Distortions also depend on the process of aggregation and the range of areas produced. The hierarchical process tends to produce a wider variation in area, showing a good fit to the Pareto distribution, whereas the areas produced by the S-mosaic process are more uniform and follow a gamma distribution. The

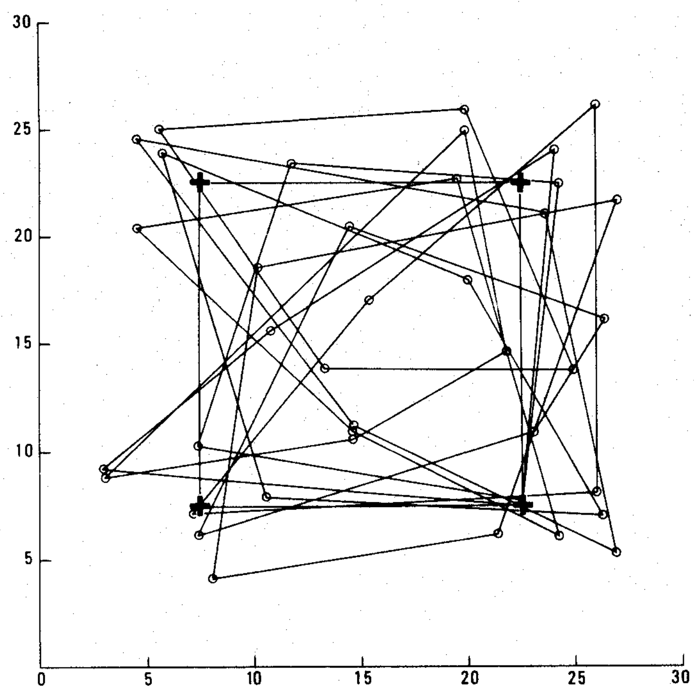

FIG. 5. Independent Replications of Hierarchical Aggregations to 150 Areas, 4 Medians

distortions produced by the S-mosaic process are generally smaller, and although the hierarchical distortion is high at $n = 150$, the S-mosaic becomes comparable only for $n < 50$. There is, of course, no general size distribution for geographical areas, but the literature contains support for both of the models used here, so they cannot be rejected as unrealistic. The size distribution produced by the S-mosaic process is approximately gamma, and has been used as a model of political and administrative divisions. Census tracts, which are frequently used as the basis for location-allocation studies, show a size distribution with a long high tail of large, suburban tracts and tend to follow the lognormal or Pareto distributions, which are produced by the hierarchical aggregation process used here (for a review of this literature see [*2*, *4*] or [*9*]. Mandelbrot [*17*] has presented a model for the Pareto distributions observed for certain geographic areas.)

4. NORMATIVE ZONING FOR MEDIAN PROBLEMS IN CONTINUOUS SPACE

Since zoning can produce considerable distortion of median solutions, this section considers the possibility of zoning in order to produce certain prescribed ends. In particular, a zoning scheme can be chosen to have minimal effect on a solution, or to have the maximum possible effect.

The hierarchical algorithm used in the previous section was modified so that joins could be chosen to satisfy certain criteria, rather than generated randomly. First, joins were made only between areas assigned to the same median, since joins across allocation boundaries more clearly affect the solution. Within this constraint, from within the area assigned to each median, the join was selected that left the net distortion of the median closest to zero. This allows joins to reduce or remove the distorting effects of previous aggregations.

The results on the 1-median solution are shown in Figure 6 for 550 areas. The figure shows the pattern of aggregation at this stage. The shaded area has not been aggregated, and so consists of single cells. The distortion of the median is effectively zero. There are systematic tendencies due to the use of a square array of base points, particularly in the long thin areas pointing to the median from each of the four prime directions. The tendency is for joins to be made first between cells near the periphery, and subtending a small angle at the median, consistent with equations (1).

The enormous number of zoning patterns available means that it is possible to satisfy a wide variety of objectives by similar methods, comparable to the AZP's (automated zoning procedures) defined by Openshaw [*20*]. But since the distortion produced by aggregation is dependent on a particular solution, there can be no method of zoning to minimize distortions for all solutions. We can

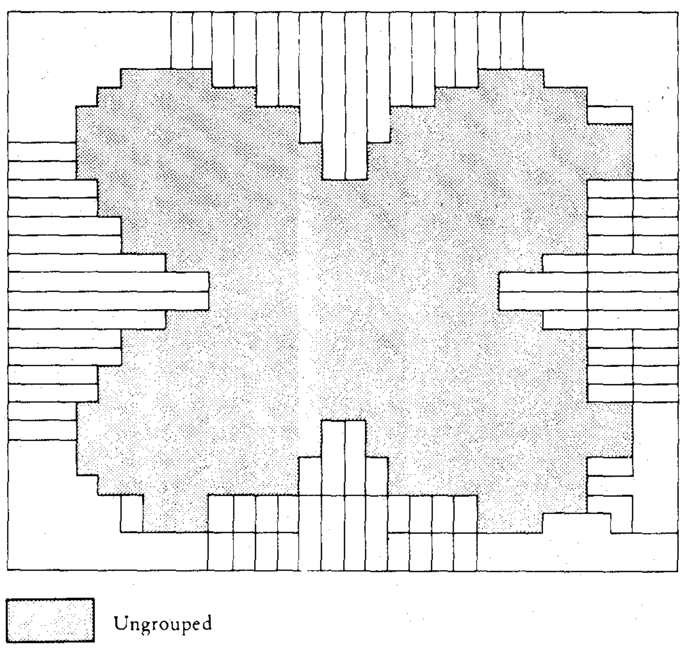

FIG. 6. Normative Aggregations to $n = 550$, 1-Median Solution

infer, however, that aggregations are relatively safe near the boundary of the study area, in regions of low weight density, and between pairs that are aligned in a direction away from the center of the study area.

5. MINIMAX PROBLEMS IN CONTINUOUS SPACE

Minimax problems are taken here to mean those in which m centers are located so as to minimize the maximum distance between demand points and their respective centers. Aggregation can only affect the solution to a minimax problem if it would result in a change in maximum distance. This can occur in two ways; in an aggregation involving the node or nodes that are at the maximum distance in a particular solution, or in an aggregation between other nodes that would place the new aggregate node at more than the maximum distance from its assigned center. The latter case may arise with pairs of nodes assigned to different centers. The minimax problem is similar to the median, then, in that in both cases the effects of aggregation are unique to particular solutions, and therefore in that no general rules for aggregation can be found. Although some aggregations are more likely to cause distortions than others, in the median case the safest aggregations are at the periphery of the study area, whereas in the minimax case they are in the center. Coverage problems are identical in this respect to minimax.

6. NETWORK AGGREGATION

We now consider the effects of aggregation on network or discrete-space problems, taking the median case first. Let there be a subset of nodes selected as medians. Then the paths along which nodes are allocated to respective medians form a set of trees. Figure 7 shows the trees for a 6-median solution in Southwestern Ontario. Nodes have been weighted by their 1976 populations, and links are those of the provincial primary highway network. All solutions described in this section were found using Maranzana [*18*] and Teitz and Bart [*24*] heuristics as implemented by Hillsman [*11*].

By analogy to the resolution of vector notation used in the earlier sections, any aggregation between nodes in the same tree cannot affect the solution, provided the aggregate node is given the sum of the aggregated weights. Aggregations of nodes in different trees of the same median may or may not cause a distortion, depending on the weights. The greater the weight "moved" from one tree to another by the aggregation, the greater the risk. The same is true of aggregations across area boundaries. The dashed lines in Figure 7 show the inter-area boundaries and also the lines separating the individual trees of each median (the "interfluves").

We can now classify every link in the network as either safe (green) or potentially unsafe (red), according to whether movement of demand along the link through aggregation could produce distortion. In this particular 6-median solution there are 88 greens and 47 reds. The proportion of reds declines with the number of medians, as one might expect, from 53 percent in the 6-median

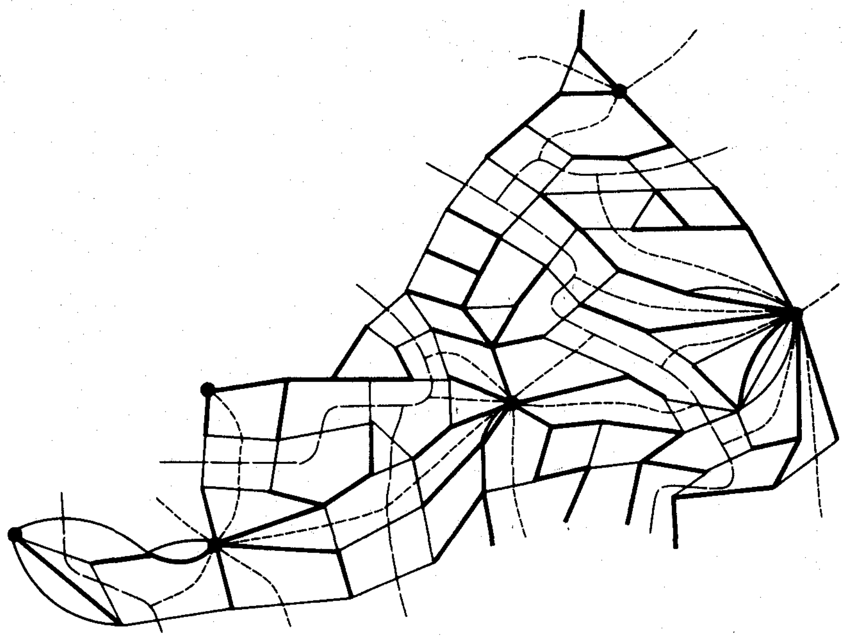

FIG. 7. 6-Median Solution for Southwestern Ontario

solution to 15 percent in the 1-median. There is some consistency in the pattern of red links over changes in the number of medians because of the tendency for heavily weighted points to become solution nodes. Links aligned in the direction of heavily weighted nodes tend to be in the same tree, and so are rarely red, and the highest scores are associated with links aligned perpendicularly to major nodes. Figure 8 shows links by the number of solutions (from 1 through 6 medians) in which each was colored red. Note that the concept of orientation must be interpreted in relation to link lengths: a link ij is "perpendicular" to a node k if $|d_{ij} - d_{ik}|/d_{jk}$ is close to 0, and "aligned in the direction of" k if the ratio is close to 1.

Of the 88 links that were green for 6 medians, 78 were green in all six solutions. These include links that form the only connection between an isolated node and the network, which must always be green. Short links are more likely to be green than long ones, as are links on the periphery of the study area.

By comparison, safety in relation to minimax solutions is much harder to identify. Aggregations between trees may cause distortion if they affect the observed maximum distance, and so may aggregations within trees. In general, only a small number of nodes will be at the observed maximum distance, and so only a small number of links are unsafe. As before, this set of links will be unique to each solution and will tend to concentrate on the periphery. This contrasts with the relative safety of the periphery in median solutions.

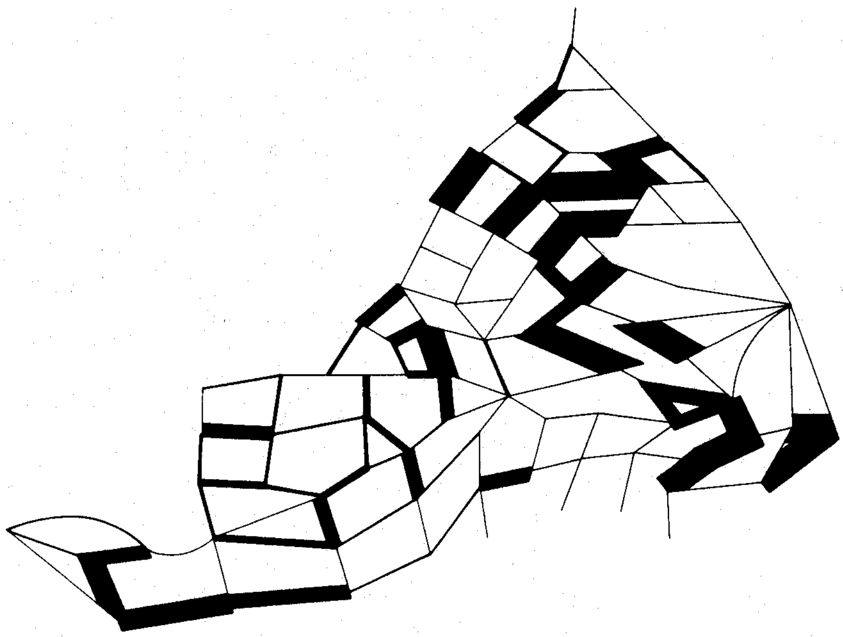

FIG. 8. Number of Median Solutions in Which Links Were Colored Red, Southwestern Ontario

Both examples considered thus far have assumed a discrete pattern of demand at the most disaggregated level. The third example is based on a continuous or near-continuous demand surface.

7. AGGREGATION FROM A CONTINUOUS SURFACE

In a recent study of fire station location in London, Ontario [27], estimates of demand were based on the observed pattern of 2,458 alarms during the calendar year 1973. The pattern can be generalized to a virtually continuous alarm density over the city, with a peak in the core area.

In the initial study the surface was collapsed onto a set of nodes by assigning each fire to its nearest node, in other words integrating the surface over the Thiessen polygons of each node. The nodes chosen were 150 major intersections of the street network in the city. The effects of this aggregation were unknown at the time, and the following section attempts to evaluate them, in two forms: first, the relationship between distortion and the number of nodes; and second, the importance of the locations of the 150 nodes. Solutions are in each case for 9 medians.

In the first experiment the number of nodes was reduced by random sampling in stages from 150 to 28, each stage being a subset of the previous stage. Figure 9 shows the migration of the medians as the number of nodes was

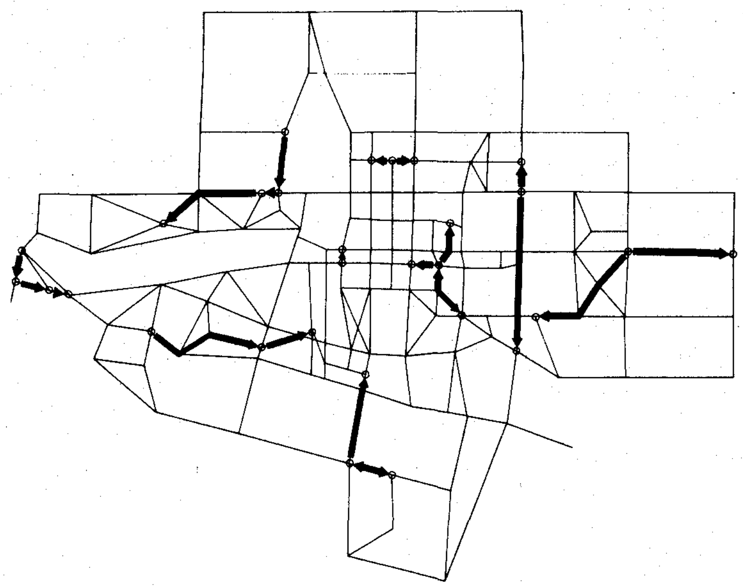

Fig. 9. Movement of Nine Medians Under Successive Elimination of Aggregation Nodes

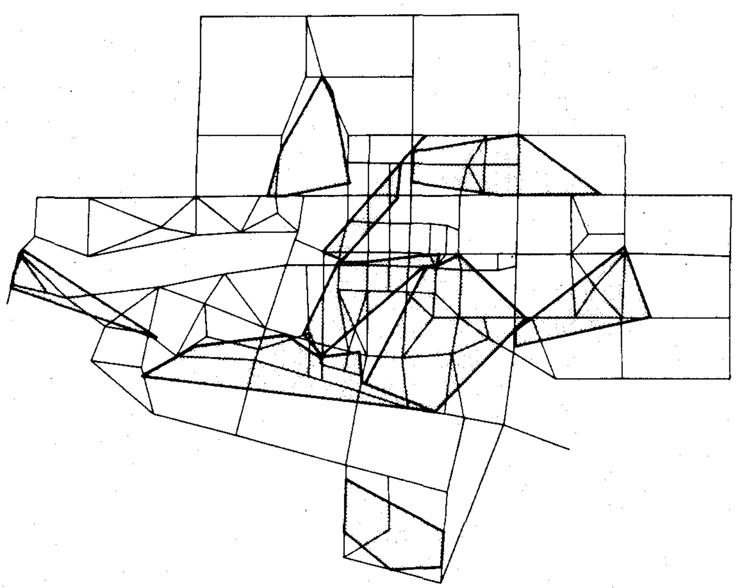

Fig. 10. Movement of Nine Medians with Redistribution of 150 Aggregation Nodes

reduced. In some cases movement is consistently in one direction; in others medians return to earlier positions as the number of nodes decreases. Because of the distances over which solutions move, the identification of medians in successive steps is to some extent ambiguous, but it has been made in such a way that no crossings of paths occur.

In the second experiment, aggregations were made to redistributed nodes. Recall that the original nodes occupied junctions on the street network. In each simulation the set of nodes was redistributed over the street network according to the following process. First, 150 links were randomly chosen from the set of 265. Then a node was placed at a random location on each link in this subset. This process ensures that the density pattern of nodes is roughly the same in the redistribution. Finally each fire was allocated to the nearest node as before.

Ten simulations were performed, and Figure 10 depicts the variation among the 9-median solutions. The shaded areas are drawn such that each contains one median in each of the ten solutions. Each area is the convex hull of ten points, except in cases where an overlap has been avoided by using a concave boundary. As in the case of Figure 9, the allocation of medians to polygons is to some extent arbitrary, but Figure 10 gives some impression of the distortion due to this form of aggregation.

8. CONCLUDING REMARKS

These experiments have shown that the effects of aggregation error on median problems are substantial. The effect on center or coverage problems is more direct and obvious; however, the strength of the effect has been expressed in terms of movement in solution locations. Goodchild [*10*] and Larson and Stevenson [*16*] have pointed out that the objective functions of median-like problems are relatively insensitive to locations. Accordingly, aggregation tends to produce much more dramatic effects on location than on the values of the objective function. But the main social advantage of location-allocation techniques, that they provide objective methods for recommending specific solutions to planning problems, seems to be lost if the specific solutions are simply the result of one particular pattern of aggregation, and not in any sense absolute.

The results in this paper show that solutions using aggregated data are open to extensive manipulation, and in fact cast some degree of doubt on the usefulness of many location-allocation models. The steps a modeler should take to minimize or evaluate the severity of the problem depend on the origin of the aggregation. In some cases aggregation is performed by the modeler to reduce the problem to manageable size; in other cases, the data were initially obtained in aggregated form and there is no access to disaggregated information. Let us take the former case, however. Denote the disaggregated data by L, the aggregated by H.

Consider a median solution in continuous space based on H. From L and the aggregation rules it is possible to compute distortion vectors for each median, which give a measure of the degree of distortion. They do not, of course, indicate the solution for L, but could be used as the basis for an iterative algorithm, in which new trial locations could be based on the magnitude and direction of each distortion vector in the previous step.

Distortion of median solutions on networks is not continuous; a certain threshold weight must be redistributed along "red" links before the solution will change, in a series of sudden jumps from node to node. The threshold condition is that the weight in one tree emanating from a solution node (see Fig. 7) exceeds the sum of the weights in all other trees from that node. Again, this condition can be identified for an H solution knowing L and the aggregation rules, and used as the basis for a measure of distortion.

The above argument is based on the assumption that solution for L is not possible. However, we find in many examples that L solution is simply much more expensive. Improved accuracy from disaggregation can then be regarded as having an intrinsic value that can be paid off against increased solution costs, as one might do in comparing exact and heuristic solutions.

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