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Spatial Choice in Location-Allocation Problems: The Role of Endogenous Attraction

Spatial choice, a voluntary form of allocation of consumers to central services, is usually conceived as affected by two factors, distance and attraction. Although usually regarded as exogenous, attraction is in turn affected by the level of use a service receives, and thus by spatial choice. This paper explores the system defined by these relationships, largely by simulation. Proposals are made concerning the initiation and perturbation of the system, and attempts are made to generalize the results. Although it is difficult to connect form with process in such a system, it is possible to identify the factors responsible for system stability.

Most of the recent progress in facility location-allocation problems has been in those areas where it is reasonable to assume a single goal for the entire system. It is in problems such as political districting [*15*], or the location of day-care centers [*9*] or of warehouses for a single corporation [*6*] that one can most readily assume that the design objectives of the system can be represented in the optimization of a single parameter, usually some form of aggregated cost.

In the simplest models both location and allocation are assumed to be under the control of the designer of the system. In the warehouse location problem, for example, the designer specifies both the locations and sizes of the warehouses, and the specific pattern of demand allocation to them; and in the political districting problem the analyst locates district boundaries, and thus implicitly makes allocations of voters to districts.

In other cases, however, it is clear that the allocation should be regarded as a matter of spatial choice on the part of the individuals responsible for demand in the system, and not placed under the control of the designer. In cases of retail store location, only the central facilities supplying service are controlled by the

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designer; the individual may choose which store to patronize in order to satisfy his own, rather than the designer's, goals. Such cases of voluntary allocation by spatial choice occur more frequently in the private sector, whereas controlled allocation is common in the provision of public sector services.

Consider the unconstrained location-allocation problem in continuous space [*4*]. The objective is to minimize the aggregate distance separating fixed demand points (individuals) and a given number of facilities. The designer must determine the locations of the facilities, and the allocation of demand.

Clearly the optimum allocation will assign each demand to its nearest facility, while each facility is optimally located with respect to its assigned demand. The solution thus satisfies the aggregate goal, and also allocates each individual to the closest facility. In this case the distinction between voluntary and controlled allocation is irrelevant if individuals behave according to the rule that each minimizes distance in making a choice.

Consider now the application of capacity constraints to the amount of demand each facility can serve (the transportation-location problem of Cooper [*5*] and see [*7*]). There will now be cases in which demand is assigned to some facility that is not the closest available. The final solution, therefore, requires a controlled allocation of individuals to centers, since in general it is impossible simultaneously to constrain the amount of demand served by each facility, and to allow each individual to behave according to an individual goal. In the central place problem, which is clearly one of voluntary allocation, some facilities must serve a greater share of the demand than others, simply because of the physical distribution of demand.

Several location-allocation models have been developed in the context of voluntary allocation. Holmes et al. [*9*] assumed that all individuals would choose the nearest center, and that demand would decrease linearly with distance traveled. Abernathy and Hershey [*1*] added an additional dimension by allowing individuals to divide their demand between several facilities in amounts that were in inverse proportion to the respective distances. But in both cases allocations were made on the basis of distance alone, on the assumption that no other criteria for individual spatial choice existed. This may be reasonable in the case of identical facilities in the public sector, but it is more usual for facilities to be differentiated along several additional dimensions. Behavior will respond both to the physical attributes of facilities, such as size, price variation, and so forth (factors that are usually subsumed under "attraction" in both gravity and preference [*13*] models of spatial behavior), and also to behavior itself in a direct feedback mechanism, since underuse and crowding are themselves important determinants of individual choice.

This paper explores some of the implications of voluntary allocation in the location-allocation field. Behavior is allowed to depend on distance, on the exogenous characteristics of each facility, and endogenously on the actual level of use experienced. The conclusions concern the stability of such systems, the problem of modeling allocations in the presence of feedback, and the question of rational decision making.

Simulations were made using a data set consisting of 49 central places, or facilities, in a small area of western India. The populations of the places were known, together with the lengths of the shortest routes between all places. For the purpose of simulation it was assumed that all rural population had been grouped into the discrete central places, giving a total population of 41,650.

The ten-median problem was solved using the methods of Maranzana [*11*] and Teitz and Bart [*16*]; places 1, 3, 8, 9, 10, 14, 16, 24, 33, and 42 form the optimal solution.

THE DYNAMIC ALLOCATION PROBLEM

Suppose that the allocation of demand in the system is controlled by a behavioral rule that combines the effects of distance with each central facility's attraction. A preference model is used arbitrarily in this paper; similar conclusions would apply to a gravity model of spatial behavior. A preference is defined for each facility at each demand point.

$$U_{ij} = U(A_j, D_{ij}),$$

where U_{ij} is the preference for facility j at demand point i; A_j is the attraction of facility j; and D_{ij} is the distance. The demand P_i is then allocated to that facility with the highest preference.

Consider first the simple additive function

$$U_{ij} = aA_j - bD_{ij},$$

indicating a preference for attractive, nearby facilities over unattractive, distant ones. b can be set to unity without loss of generality. When a is 0, choice is determined by distance alone and individuals will allocate themselves to the nearest facility. The results for the ten places of the ten-median solution are shown in the first column of Table 1.

The attraction of each place is conceived as having two components, one exogenous and constant, the other dependent on the demand allocated to each

TABLE 1

Equilibrium Solutions for Varying Additive Choice Rules

		values of parameter a							
Center	Nearest Center Allocation	0.0006	0.0007–0.0009	0.0010	0.0012	0.0014	0.0016	0.0018	>0.0018
16	7600	7600	6600	6100	6900	6500	6500	5000	0
42	2900	2900	2900	2900	2900	2900	2400	1700	0
1	1200	1200	1200	1200	1200	1200	1200	1200	0
10	2900	2900	2900	2900	2900	2900	2900	2900	0
33	10850	10850	11850	12350	12350	12750	12750	13450	0
3	1500	1500	1500	1500	1500	1500	1500	1000	0
8	3100	3100	3100	3100	2300	2300	2300	2300	0
9	3400	3400	3400	3400	3400	3400	3400	3400	0
14	7700	7700	7700	7700	7700	7700	8700	10700	41650
24	500	500	500	500	500	500	0	0	0

place. The latter relationship might operate in several ways. In the short term, a central facility can be made less attractive by overuse, or crowding, and possibly by underuse. A similar effect can occur more slowly through a price mechanism; a low use level for a supermarket can make goods more expensive, which can lead to higher prices, and hence lower attraction. Much longer-term effects can be recognized also; a high use level can lead to the physical expansion of facilities, and thus to a higher attraction.

These effects can be summarized by writing attraction as a function of total allocated demand:

$$A_j = A\left(\sum_i I_{ij}\right); \qquad \sum_j I_{ij} = P_i,$$

where I_{ij} is the demand allocated from i to j. When coupled with the behavioral rule, this equation defines a dynamic system whose state is determined by the vector of attraction values A_j. It is convenient to think of a state space with axes defined by the A_j, so that any system state can be identified with a point in state space. For every such point there corresponds a set of allocations I_{ij}, and hence a set of total demands $\Sigma_i I_{ij}$. When the A_j generated by these demands are equal to the A_j that generated the demands, the system is said to be in equilibrium.

There are algebraic parallels between this model and that of Lakshmanan and Hansen [*10*]. In a retail context $\Sigma_i I_{ij}$ is related to expected sales at j, and A_j has frequently been equated with retail floor area. Lakshmanan and Hansen argued that shopping centers should be planned for a certain level of annual sales per unit of floor area, i.e.

$$A_j = K \sum_i I_{ij}.$$

The floor areas A_j were defined exogenously, and then applied in a gravity model to predict $\Sigma_i I_{ij}$. The authors were thus able to select that set of locations and floor areas that gave the most "balanced" set of sales per unit floor area.

BOUNDARY CONDITIONS

Consider the following example. A number of facilities are to be located to serve a dispersed population. The pattern of service areas that will result can be described by a behavioral rule, which depends on an attraction level for each facility, which is in turn related to levels of use at each facility. When the facilities are first opened, the population cannot predict use levels and so bases its choices on distance and the physical component of attraction alone. This establishes use levels, which in turn modifies behavior. New use levels are established, and the cycle iterates until an equilibrium is reached.

The time taken to reach equilibrium, the relaxation time of the system, will vary depending on the example. In some cases use levels may be communicated through the population without intervening visits to the facility. In other cases the relaxation time will depend on the frequency of visits, and in others on the time required for the physical expansion of facilities. A system of service towns has an extremely long relaxation time, since attraction is related to the town population and the number of retail establishments, which responds very slowly to increasing retail expenditure.

Table 1 shows the results of iterating from this null boundary condition for various values of the parameter a in the trip utility function. Attraction was equated with total demand, which is appropriate in the central place example [2], and in other cases where facilities can be expanded to meet increasing demand. In each case the iterations converged to an equilibrium point; when a exceeded 0.0018 one facility expanded to capture all of the demand in the system. Below $a = 0.0007$ the initial null condition of nearest-center allocation was also the equilibrium state.

PERTURBATIONS

In general a state space will contain more than one equilibrium point for a given parameterization of the model. Shocks and perturbations to the system, which temporarily change one or more attraction values, may result in a return to the same equilibrium or a shift to a new one. Advertising represents one possible example of a perturbation. A temporary rise in one facility's attraction will modify spatial choice patterns and may increase the total demand at the facility. If this is sufficiently large, the attraction will continue to increase until a new equilibrium state is reached. Thus the state space can be partitioned into a set of domains around each equilibrium point, such that if a perturbation occurs to a state within a domain, the system relaxes to the respective equilibrium. Other sources of perturbation include temporary closure, diversification of service, and facility expansion. It is also possible for the parameters of the system to change, as when the payoff between distance and attraction is altered by changing modes of transportation. The new parameters will change the geometry of the state space, and the system will relax towards a new equilibrium. But if the parameters change continuously, a system with a long relaxation time may never reach equilibrium.

Perturbations were made to explore the geometry of the state space for $a = 0.0006$. Each attraction value was multiplied by a log-normally distributed random number using the method of Box and Muller [3]

$$\acute{A}_j = A_j \exp\left(\sigma\sqrt{-2\log R_1}\ \sin 2\pi R_2\right),$$

where R_1 and R_2 are independent uniformly distributed random numbers in the interval $(0, 1)$, and σ is a parameter representing the average shock magnitude. The average absolute value of $\log(\acute{A}_j / A_j)$ is $\sqrt{2/\pi}\ \sigma$.

The results of one hundred perturbations and subsequent relaxations are shown in Table 2 for $\sigma = 0.25$. Ten equilibrium points were found, of which the unperturbed solution is by no means the most common. For $\sigma = 0.25$, the geometric mean perturbation factor is 1.22.

TABLE 2

Equilibrium States for Perturbation Parameter 0.25

Center	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*
16	7600	6600	8300	6600	8300	7300	7600	7300	13600	13600
42	2900	2900	2900	2900	2900	2900	2900	2900	2400	2400
1	1200	1200	1200	1200	1200	1200	1200	1200	1200	1200
10	2900	2900	3900	3900	2900	2900	3900	3900	3900	2900
33	10850	11850	10850	11850	10850	11850	10850	11850	10350	10350
3	1500	1500	1500	1500	1500	1500	1500	1500	1500	1500
8	3100	3100	3100	3100	3100	3100	3100	3100	2300	2300
9	3400	3400	2400	2400	3400	3400	2400	2400	2400	3400
14	7700	7000	7000	7700	7000	7000	7700	7000	3500	3500
24	500	500	500	500	500	500	500	500	500	500
Frequency	13	23	16	13	16	7	4	5	1	2

DISCUSSION

The number and sizes of domains and the locations of equilibria are affected by a number of parameters, including the form of the U and A functions, the geographical distribution of demand, and the locations of facilities. Yet it is difficult to establish relationships between the form of the system, in other words the geometry of its state space, and the processes or parameters that define it. One set of parameters can lead to a number of equilibrium states, and one equilibrium can result from a number of different sets of parameters.

It is much easier to generalize the factors responsible for overall system instability. In the previous examples, attraction was allowed to rise indefinitely with increasing demand, and to decrease to zero when no demand was allocated ($A_j = \Sigma_i I_{ij}$). Consider the function $A_j = 5{,}000 + 0.5\Sigma_i I_{ij} - 0.00005(\Sigma_i I_{ij})^2$, which has the form of an inverted parabola. Attraction is finite at zero demand, rises to a maximum at an optimum level of demand, and then drops when further demand might result in crowding or diseconomies of scale. The result is a greater resistance to perturbation in the system, and one hundred simulations at $\sigma = 0.25$ produced only two equilibria, thirty-eight and sixty-two times respectively.

A measure of overall instability must respond to the number of domains in the state space, and also to their relative size and proximity. For these reasons the information statistic

$$H = -\sum_i P_i \log P_i,$$

where P_i is the proportion of perturbations that result in a relaxation to equilibrium state i, is proposed as an adequate measure of instability. H increases with the number of domains, and for a given number is maximum when all domains are equally likely to be probed by a perturbation. There are interesting parallels with the use of H as a measure of stability in biological systems [*8*, *12*].

The system of Table 2, where attraction was equated to demand, has an instability of 2.04, whereas the more stable inverted parabola gives a value of 0.66. The form of the utility function has a similar effect on instability. For a less than 0.0006, only one domain is observed and H is 0. At $a=0.0006$, H is 2.04, and at high values of a, when each of the ten facilities has a roughly equal chance of capturing the market, H approaches the theoretical maximum of log(10) or 2.30.

H unfortunately depends to some extent on the perturbation parameter σ, since the system is more resistant to small shocks than large ones. Because all perturbations are made from a single state, σ controls the distribution of perturbed states and thus the number and relative frequency of equilibria found. 0.25 was chosen because higher values failed to expose any new equilibria.

CONCLUDING REMARKS

Spatial behavior is usually modeled in terms of fixed, exogenously defined parameters. Distance is a purely physical quantity, and its counterpart, attraction, is usually represented as some multivariate measure of the characteristics of each available offering of the good or service. This paper has added a dynamic element, by suggesting that attraction is often related to the use of a facility, and thus creates a loop back to the spatial behavior itself. The resulting model is potentially capable of describing a number of time-dependent spatial phenomena.

Only the allocation problem has been considered. Location raises a number of further topics, since it is possible to design the physical attraction of each facility, to influence system stability, and to control the possibility that any facility will eventually become either dominant or redundant.

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