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# A Hierarchical Spatial Data Structure for Global Geographic Information Systems

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**Hierarchical spatial data structures offer the distinct advantages of data compression and fast access, but are difficult to adapt to the globe. Following Dutton, we propose projecting the globe onto an octahedron and then recursively subdividing each of its eight triangular faces into four triangles. We provide procedures for addressing the hierarchy and for computing addresses in the hierarchical structure from latitude and longitude, and vice versa. At any level in the hierarchy the finite elements are all triangles, but are only approximately equal in area and shape; we provide methods for computing area and for finding the addresses of neighboring triangles.** 

## INTRODUCTION

Hierarchical spatial data structures (HSDSs) such as the quadtree and octtree (see, for example, [4]) have been adopted in numerous geographic information systems and spatial databases. They offer advantages in data compression and sampling efficiency, since the depth of the tree, and thus the density of information, can be varied from one area to another in response to the variability of the phenomenon being represented. Numerous processes operate faster on HSDSs, particularly various forms of spatial search. The address of a cell in an HSDS embeds both of its spatial coordinates and thus effectively compresses two dimensions into one [5]. Cells lower in the tree have longer addresses, and the length of an address is therefore a direct measure of spatial resolution. This has led to the suggestion that HSDSs offer a powerful solution to the problems of accuracy in spatial databases, since the spatial resolution of a position can be determined directly from the length of its spatial address [2, 6].

Three properties of quadtree and octtree implementations of HSDSs are of particular interest in this paper: (1) at any level, the cells are equal in area; (2) at any level, cells are equal in shape; and (3) the data structure correctly encodes the adjacency relationships between cells. The value of an HSDS for analysis and modeling would clearly be reduced without these properties, particularly in modeling based on finite elements. Unfortunately it has proven difficult to find a method of hierarchically subdividing the earth's surface so that these properties are retained. Many global databases have been based on rectangular cells superimposed on simple cylindrical projections such as Mercator's or the cylindrical equidistant projection. However, although these schemes may achieve one of our required properties (as a conformal projection, the Mercator projection achieves property (1)), we note that it is well known that no projection of the earth onto a plane can satisfy both of properties (1) and (2). Moreover any cylindrical projection must violate property (3) because of the interruption at the poles. A method based on a cylindrical equal area projection was proposed by Tobler and Chen [7]; cells at a given level have equal area, but unequal shape.

An HSDS called *Triacon* or *Quaternary Triangular Mesh* (*QTM*) was suggested by Dutton [1, 2]. In this paper we follow Dutton's approach in first projecting the Earth onto an octahedron and then recursively subdividing each of the eight triangular faces of the octahedron into four triangles. Each level of the hierarchy after the first thus contains four times as many triangular cells or elements as the previous level. We simplify Dutton's approach in our numbering of the triangles, in order to obtain an addressing system that provides easy transformation to and from latitude and longitude. Our scheme satisfies property (3), and although properties (1) and (2) are only approximately satisfied, each triangular cell has an area that can be computed from a simple expression.

An approach somewhat similar to ours has been proposed by Fekete on the basis of recursive subdivision of the triangular faces of an icosahedron [8–10]. Fekete's scheme has the advantage that the larger number of faces of the basic Platonic solid (20 rather than 8) produces a greater uniformity of area at any given level of subdivision. However, the algorithms associated with our octahedral scheme tend to be simpler, and the six vertices of the base octahedron align with the poles and equator.

The discussion is organized as follows. We first de-



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scribe the coordinate systems used to develop the properties of the proposed global HSDS. Subsequent sections develop the transformations between coordinate systems, particularly between cell address and latitude/longitude. Section 4 discusses the calculation of cell area, and Section 5 and Section 6 present an algorithm for finding the neighbors of a cell and the average data file storage distance, which corresponds to the expected cost of transition from one data cell to its neighbors.

## 1. COORDINATE SYSTEMS

In our proposed scheme, the entire earth is described by an octahedron. One-quarter of each hemisphere is represented by an equilateral triangle and is then decomposed. In order for the hierarchy to be symmetrical and isohedral, i.e., all cells are congruent and every cell can be mapped onto other cells through translation, reflection, rotation or a combination of these, the triangle is subdivided into four smaller equilateral triangles and each of them is further subdivided recursively until a required level is reached. When the four triangles are decomposed from their ancestor triangle, they are labeled 0, 1, 2, and 3. There are 24 possible distinct schemes of labeling. Moreover while the initial subdivision occurs with the northern hemisphere triangles standing on their bases ("upward"), the southern hemisphere triangles stand on vertices ("downward"). In subsequent iterations, triangles in both upward and downward orientations must be subdivided in both hemispheres. If this orientation of triangles is considered, there are 48 possible labeling schemes. In order to limit the complexity of the addressing and conversion algorithms, we use the following method in every recursive decomposition: (1) the center triangle is labeled cell 0; (2) the triangle vertically above (below) the central triangle is labeled cell 1; and (3) the triangles below (above) and left and right of triangle 0 are labeled cells 2 and 3, respectively. Note that the terms in the parentheses are used when the triangle being subdivided stands on a vertex (downward).

The initial representation of the globe as eight triangles is termed the level 0 subdivision; after $j$ further subdivisions of each triangle we reach level $j$ of the HSDS. Thus at level $j$, there are $8 \times 4^j$ cells. For much of the discussion in this paper the level 0 subdivision is ignored, and we refer simply to the recursive subdivision of one-quarter hemisphere. Figure 1a is a triangle decomposed to level 4, with each cell identified by its address, which consists of four base-4 digits, identifying the triangles selected at each level of subdivision. The full address including level 0 would require an initial base-8 digit. Figure 1b shows the decimal address of cells in the triangular decomposition. Figure 1c shows the ordering of cells and emphasizes the consistent choice of the left cell as cell 2 at every level, irrespective of whether the triangle is upward or downward.

In this study, we use the following coordinate systems:

(1) Positions on the globe are referenced by latitude $\phi$ and longitude $\lambda$.

(2) Each of the eight isosceles triangles of the level 0 octahedron contains one-quarter hemisphere. Locations within each triangle are identified by Cartesian coordinates $x$ and $y$, with respect to an origin in the lower left corner. The triangles are numbered 0 through 3 in the northern hemisphere and 4 through 7 in the southern, in both cases in anticlockwise order when viewed from the north pole (see Fig. 2).

The triangles vertices are assumed to lie at (0, 0), and $(2^n, 0)$ and $(2^{n-1}, 2^{n-1}\sqrt{3})$ in the $(x, y)$ coordinate system, where $n$ is the highest level of subdivision. We assume that $x$ depends linearly on longitude for a given latitude, and that $y$ depends linearly on latitude. Figure 3 shows the relationships between latitude and longitude and $(x, y)$ schematically. The left, right, and bottom edges of the triangle in Fig. 3 can be described by the following equations:

| | | | |
|---|---|---|---|
| Left edge: | $y = \sqrt{3}x$ | or $\lambda = 0$ | (1-1) |
| Right edge: | $y = (2^n - x)\sqrt{3}$ | or $\lambda = \frac{\pi}{2}$ | (1-2) |
| Bottom edge: | $y = 0$ | or $\phi = 0.$ | (1-3) |

From Fig. 3 and expressions (1-1) to (1-3), we have the following expression for the relation between $y$ and $\phi$:

$$\phi = \frac{\pi}{2^n\sqrt{3}} y \quad \text{or} \quad y = \frac{2^n\sqrt{3}}{\pi}\phi. \tag{1-4}$$

From the point $p(x, y)$ in Fig. 3, we have

$$x = x_1 + x', \quad \text{where} \quad x_1 = \frac{1}{\sqrt{3}} y$$

and $x'$ is the horizontal distance from $p(x, y)$ to the left edge of the triangle. Since the distance between $x_2$ and $x_1$ corresponds to the maximum longitude difference at latitude $\phi = \pi y/(2^n\sqrt{3})$, which is also defined as $\pi/2$, we have

$$\frac{x'}{x_2 - x_1} = \frac{2\lambda}{\pi}.$$

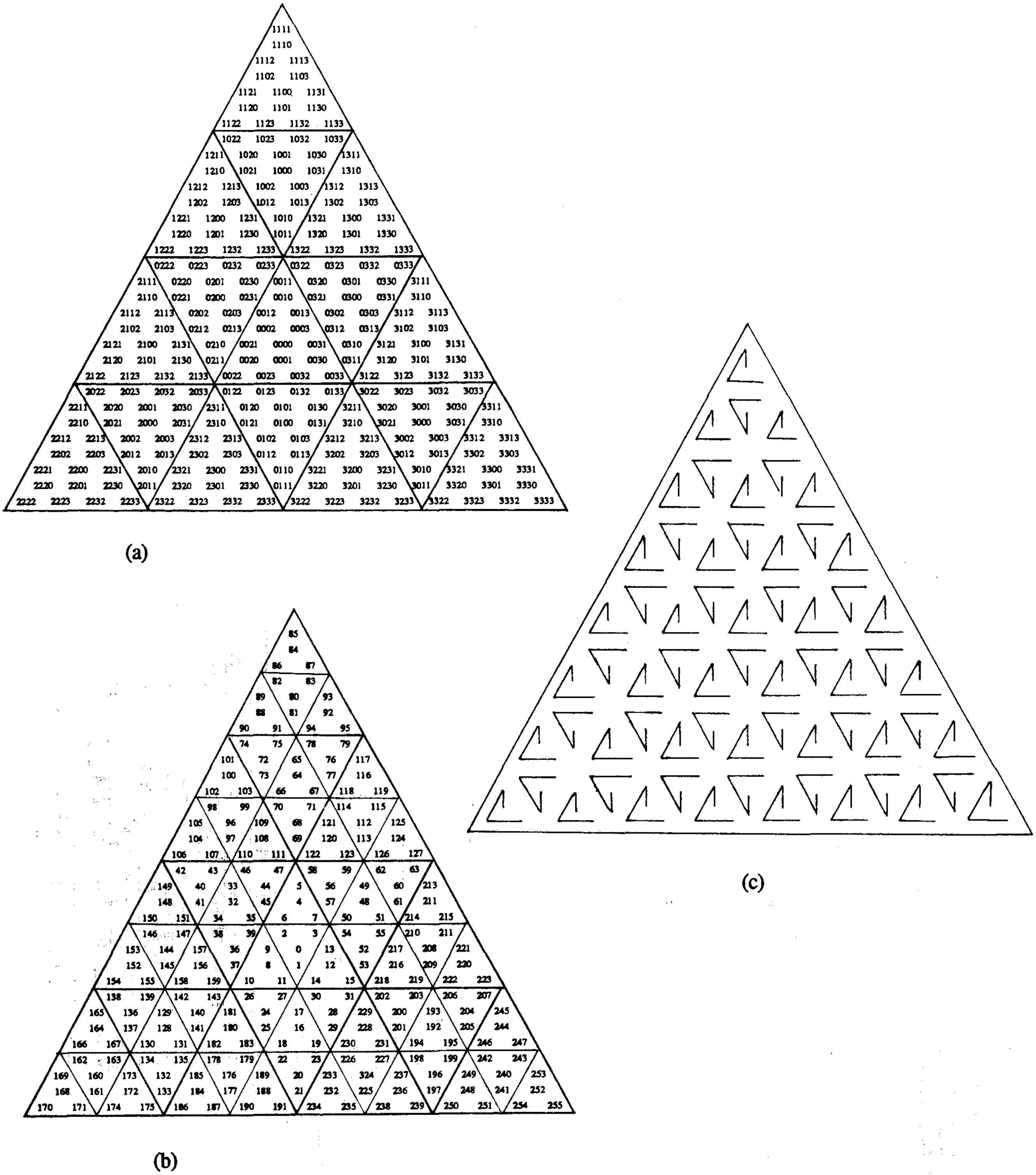

FIG. 1. The triangular data structure applied to a quarter hemisphere, showing (a) quaternary addressing, (b) decimal addressing, and (c) the ordering of level-4 triangles.

From expressions (1-1) and (1-2), we have

$$x_2 = 2^n - \frac{y}{\sqrt{3}}.$$

Then

$$x' = \frac{2\lambda}{\pi}(x_2 - x_1) = \frac{2^{n+1}\lambda}{\pi}\left(1 - \frac{2^{1-n}}{\sqrt{3}}\,y\right)$$

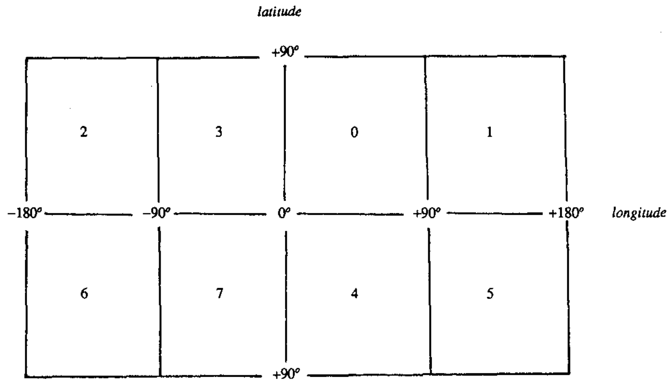


FIG. 2. Initial octahedral decomposition of the Earth at level 0.

and

$$x = x_1 + x' = \frac{y}{\sqrt{3}} + \frac{2^{n+1}\lambda}{\pi}\left(1 - \frac{2^{1-n}}{\sqrt{3}}y\right)$$

$$= \frac{2^n}{\pi}\left[\phi + 2\lambda\left(1 - \frac{2}{\pi}\phi\right)\right].$$

The expressions for transformation of longitude $\lambda$ and latitude $\phi$ to $x$ and $y$ in the triangle are

$$x = \frac{2^n}{\pi}\left[\phi + 2\lambda\left(1 - \frac{2}{\pi}\phi\right)\right], \quad y = \frac{2^n\sqrt{3}}{\pi}\phi \qquad (1\text{-}5)$$

or

$$\lambda = \frac{\pi}{2^{n+1}}\frac{\sqrt{3}x - y}{\sqrt{3} - 2^{1-n}y}, \quad \phi = \frac{\pi}{2^n\sqrt{3}}y. \qquad (1\text{-}6)$$

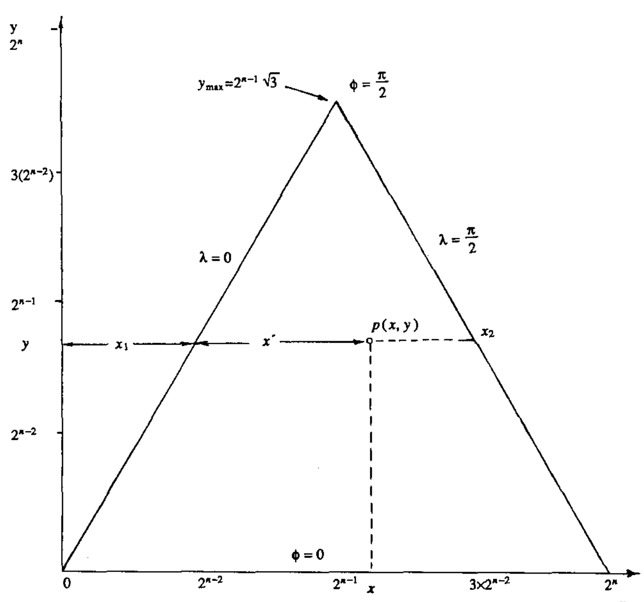


FIG. 3. Relationships between the coordinate systems.

**TABLE 1**
**Basic Geometric Parameters of Decomposed Triangles**

| Level | Edge length | Height | Center to left side | Center to bottom | Center to top |
|---|---|---|---|---|---|
| 0 | $2^n$ | $2^{n-1}\sqrt{3}$ | $2^{n-1}$ | $\frac{2^{n-1}}{\sqrt{3}}$ | $\frac{2^n}{\sqrt{3}}$ |
| 1 | $2^{n-1}$ | $2^{n-2}\sqrt{3}$ | $2^{n-2}$ | $\frac{2^{n-2}}{\sqrt{3}}$ | $\frac{2^{n-1}}{\sqrt{3}}$ |
| 2 | $2^{n-2}$ | $2^{n-3}\sqrt{3}$ | $2^{n-3}$ | $\frac{2^{n-3}}{\sqrt{3}}$ | $\frac{2^{n-2}}{\sqrt{3}}$ |
| . . . | . . . | . . . | . . . | . . . | . . . |
| $j$ | $2^{n-j}$ | $2^{n-j-1}\sqrt{3}$ | $2^{n-j-1}$ | $\frac{2^{n-j-1}}{\sqrt{3}}$ | $\frac{2^{n-j}}{\sqrt{3}}$ |

The parameters of the triangles at different levels are shown in Table 1. Table 2 shows the lengths of edges of the triangles at different levels of decomposition. At the 20th level, the edges of triangles are less than 10$m$, and 20 quaternary digits or 40 binary digits (approximately 12 decimal digits) are required for addressing.

## 2. CONVERSION OF TRIANGLE ADDRESS TO CARTESIAN COORDINATES

The addresses of vertices can be calculated by using the parameters listed in Table 1. For triangles decomposed to the $k$th level, the *triangle address* is represented by $k$ quaternary numbers

$$a_1, a_2, a_3, \ldots, a_k, \qquad (2\text{-}1)$$

where $1 \leq k \leq n$, plus the base-8 digit $a_0$. In this section we consider the problem of determining the Cartesian

**TABLE 2**
**Length of Triangle Edges at Increasing Levels of Subdivision of a Spheroid with Radius 6378 km along Equator and 6356 km along Meridian**

| Level | Degree | Along equator (longitude) | Along meridian (latitude) |
|---|---|---|---|
| 0 | 90° | 10018.5380 km | 9983.8912 km |
| 2 | 22°30′ | 2504.6345 km | 2495.9953 km |
| 4 | 5°37′30″ | 626.1586 km | 623.9988 km |
| 6 | 1°24′22.5″ | 156.5397 km | 155.9997 km |
| 8 | 21′5.625″ | 39.1349 km | 38.9999 km |
| 10 | 5′16.40625″ | 9.7837 km | 9.7500 km |
| 12 | 1′19.1015″ | 2.4459 km | 2.4376 km |
| 14 | 19.77539″ | 611.5 m | 609.4 m |
| 16 | 4.9438″ | 153.9 m | 152.3 m |
| 18 | 1.23596″ | 38.2 m | 38.1 m |
| 20 | 0.3089904″ | 9.55 m | 9.525 m |

**TABLE 3**
**Relative Distances of Successor Triangles' Centroids to the Centroid of Their Ancester Triangle**

| Level | $O_{0,j} - O_{j-1}$ $(\Delta X, \Delta Y)$ | $O_{1,j} - O_{j-1}$ $(\Delta X, \Delta Y)$ | $O_{2,j} - O_{j-1}$ $(\Delta X, \Delta Y)$ | $O_{3,j} - O_{j-1}$ $(\Delta X, \Delta Y)$ |
|---|---|---|---|---|
| 1 | (0, 0) | $\left(0, \frac{2^{n-1}}{\sqrt{3}}\right)$ | $\left(-2^{n-2}, -\frac{2^{n-2}}{\sqrt{3}}\right)$ | $\left(2^{n-2}, -\frac{2^{n-2}}{\sqrt{3}}\right)$ |
| 2 | (0, 0) | $\left(0, \alpha\frac{2^{n-2}}{\sqrt{3}}\right)$ | $\left(-2^{n-3}, -\alpha\frac{2^{n-3}}{\sqrt{3}}\right)$ | $\left(2^{n-3}, -\alpha\frac{2^{n-3}}{\sqrt{3}}\right)$ |
| . . . | . . . | . . . | . . . | . . . |
| $j$ | (0, 0) | $\left(0, \alpha\frac{2^{n-j}}{\sqrt{3}}\right)$ | $\left(-2^{n-j-1}, -\alpha\frac{2^{n-j-1}}{\sqrt{3}}\right)$ | $\left(2^{n-j-1}, -\alpha\frac{2^{n-j-1}}{\sqrt{3}}\right)$ |

coordinates of the centroid of a triangle with given triangle address, ignoring the level-0 digit. The coordinates of the triangle's vertices can be determined from the parameters listed in Table 1 and from knowledge of the triangle's orientation.

With the triangle cell ordering shown in Fig. 1, there are the following relations:

(1) Let

$$NZ_k = \sum_{i=1}^{k-1} [a_i = 0] = \sum_{i=1}^{k-1} [\bar{a}_{i1} \cap \bar{a}_{i2}] \qquad (2\text{-}2)$$

denote the number of zeros in $a_1$ to $a_{k-1}$. $a_{i1}$ and $a_{i2}$ are two binary digits representing each quaternary digit $a_i$. $\bar{a}_{i1}$, $\bar{a}_{i2}$ are logical negative of (or *NOT*) $a_{i1}$, $a_{i2}$. $(a_{i1}, a_{i2})$ = (0, 0), (0, 1), (1, 0), (1, 1) correspond to $a_i$ = 0, 1, 2, 3, respectively.

If the level 0 triangle is upward, then the $k$th level triangles with address

$$A = a_1, a_2, a_3, \ldots, a_k$$

are upward if $NZ_k$ is even and downward if $NZ_k$ is odd. $NZ_k$ is zero when $k = 1$.

We can readily generalize to include the base-8 digit representing the initial octahedral decomposition at level 0. If the base-8 digit is represented as three binary digits, then the level zero triangle is upward if the first digit is 0 and downward if it is 1. The generalized definition of $NZ_k$ is

$$NZ_k = a_{00} + \sum_{i=1}^{k-1} [\bar{a}_{i1} \cap \bar{a}_{i2}],$$

where $a_{00}$ is the first bit of the binary representation of the base-8 digit.

(2) For the triangles in the $k$th level with the same ancestor triangle, i.e., having identical $a_1$ to $a_{k-1}$,

1. the triangle with $a_k = 1$ is a complex conjugate or a reflection of the triangle with $a_k = 0$;

2. the triangles with $a_k = 2$ and $a_k = 3$ are the left-down (or left-up) and right-down (or right-up) translation of the triangle with $a_k = 1$, respectively.

(3) The centroid coordinates of the 0 level (original) triangle are

$$(X_0, Y_0) = \left(2^{n-1}, \frac{2^{n-1}}{\sqrt{3}}\right). \qquad (2\text{-}3)$$

(4) The relative distance of the triangle centroids at the $k$th level $O_{k,0}$, $O_{k,1}$, $O_{k,2}$, and $O_{k,3}$ from the centroid of their ancestor at the $(k - 1)$th level $O_{k-1,0}$ are (see Table 3)

$$O_{k-1,0} - O_{k,0}: \quad (\Delta X_{k,0}, \Delta Y_{k,0}) = (0, 0) \qquad (2\text{-}4a)$$

$$O_{k-1,0} - O_{k,1}: \quad (\Delta X_{k,1}, \Delta Y_{k,1}) = \left(0, \alpha\frac{2^{n-k}}{\sqrt{3}}\right) \qquad (2\text{-}4b)$$

$$O_{k-1,0} - O_{k,2}: \quad (\Delta X_{k,2}, \Delta Y_{k,2}) = \left(-2^{n-k-1}, -\alpha\frac{2^{n-k-1}}{\sqrt{3}}\right) \qquad (2\text{-}4c)$$

$$O_{k-1,0} - O_{k,3}: \quad (\Delta X_{k,3}, \Delta Y_{k,3}) = \left(2^{n-k-1}, -\alpha\frac{2^{n-k-1}}{\sqrt{3}}\right), \qquad (2\text{-}4d)$$

where

$$\alpha = (-1)^{NZ_k}.$$

The triangle address $A = a_1, a_2, \ldots, a_k$ can be converted to Cartesian coordinates by the expressions

$$X_k = 2^{n-1} + \sum_{i=1}^{k} [(-1)^{a_{i2}+1} a_{i1} 2^{n-i-1}]$$

$$= \left[2^k + \sum_{i=1}^{k} [(-1)^{a_{i2}+1} a_{i1} 2^{k-i}]\right] 2^{n-k-1} \qquad (2\text{-}5a)$$

$$Y_k = \frac{2^{n-1} + \sum_{i=1}^{k} [(-1)^{NZ_i + a_{i1}}[2(\bar{a}_{i1} \cap a_{i2}) + a_{i1}]2^{n-i-1}]}{\sqrt{3}}$$
$$= \frac{2^{k} + \sum_{i=1}^{k} [(-1)^{NZ_i + a_{i1}}[2(\bar{a}_{i1} \cap a_{i2}) + a_{i1}]2^{k-i}]}{\sqrt{3}} 2^{n-k-1}. \quad \text{(2-5b)}$$

Coordinates of the three vertices of the triangles can be calculated from

Top vertex: $X_{k\text{-Top}} = X_k$ (2-6a)

$$Y_{k\text{-Top}} = Y_k + (-1)^{NZ_k} \frac{2^{n-k}}{\sqrt{3}} \quad \text{(2-6b)}$$

Left vertex: $X_{k\text{-Left}} = X_k - 2^{n-k-1}$ (2-7a)

$$Y_{k\text{-Left}} = Y_k + (-1)^{NZ_k+1} \frac{2^{n-k-1}}{\sqrt{3}} \quad \text{(2-7b)}$$

Right vertex: $X_{k\text{-Right}} = X_k + 2^{n-k-1}$ (2-8a)

Right vertex: $Y_{k\text{-Right}} = Y_k + (-1)^{NZ_k+1} \dfrac{2^{n-k-1}}{\sqrt{3}}.$ (2-8b)

EXAMPLE 1. Find the Cartesian coordinates of the centroid of triangle $A = 3023$:

$$A = 3023 = 11 \quad 00 \quad 10 \quad 11$$

or

$$a_{11} = 1, a_{12} = 1; \quad a_{21} = 0, a_{22} = 0;$$
$$a_{31} = 1, a_{32} = 0; \quad a_{41} = 1, a_{42} = 1;$$
$$X = [2^4 + 1 \times (-1)^{1+1}2^3 + 0 \times (-1)^{0+1}2^2 + 1 \times (-1)^{0+1}2^1 + 1 \times (-1)^{1+1}2^0]2^{n-5}$$
$$= [2^4 + 2^3 - 2^1 + 2^0]2^{n-5} = 23 \times 2^{n-5}$$
$$Y = [2^4 + 1 \times (-1)^{0+1}2^3 + 0 \times (-1)^{0+0}2^2 + 1 \times (-1)^{1+1}2^1 + 1 \times (-1)^{1+1}2^0] \frac{2^{n-5}}{\sqrt{3}}$$
$$= [2^4 - 2^3 + 2^1 + 2^0] \frac{2^{n-5}}{\sqrt{3}} = 11 \times \frac{2^{n-5}}{\sqrt{3}}.$$

EXAMPLE 2. Find the Cartesian coordinates of the centroid of triangle $A = 1033$:

$$A = 1003 = 01 \quad 00 \quad 00 \quad 11$$

or

$$a_{11} = 0, a_{12} = 1; \quad a_{21} = 0, a_{22} = 0;$$
$$a_{31} = 0, a_{32} = 0; \quad a_{41} = 1, a_{42} = 1;$$
$$X = [2^4 + 0 \times (-1)^{1+1}2^3 + 0 \times (-1)^{0+1}2^2 + 0 \times (-1)^{1}2^{0+1} + 1 \times (-1)^{1+1}2^0]2^{n-5}$$
$$= [2^4 + 2^0]2^{n-5} = 17 \times 2^{n-5}$$
$$Y = [2^4 + 2 \times (-1)^{0+0}2^3 + 0 \times (-1)^{0+0}2^2 + 0 \times (-1)^{1+0}2^1 + 1 \times (-1)^{2+1}2^0] \frac{2^{n-5}}{\sqrt{3}}$$
$$= [2^5 - 2^0] \frac{2^{n-5}}{\sqrt{3}} = 31 \times \frac{2^{n-5}}{\sqrt{3}}.$$

## 3. CONVERSION OF CARTESIAN COORDINATES TO TRIANGLE ADDRESS

For implementation of the triangular tessellation data structure in a global geographic information system, it is necessary to convert the coordinates of the Earth to the triangle address. Since longitude and latitude can be directly represented by Cartesian coordinates, the problem can be reduced to conversion of Cartesian coordinates into the triangle address. A recursive approximation algorithm for conversion of Cartesian coordinates to triangle address is derived below.

Let an equilateral triangle be divided into four triangles and let the centroid of the parent triangle be denoted by $O_{j-1,0}$; the centroids of four son triangles are denoted $O_{j,0}$, $O_{j,1}$, $O_{j,2}$, $O_{j,3}$ as shown in Fig. 4. We have $O_{j-1,0} = O_{j,0}$ and the distance between the centroid of the parent triangle and the centroids of the other three son triangles can be calculated by expressions (2-4a) to (2-4d).

If $p(x, y)$ is an arbitrary point with Cartesian coordinates $(x, y)$ and is inside the parent triangle, the point $p(x, y)$ will be a point in the $k$th son triangle if $p(x, y)$ is closest

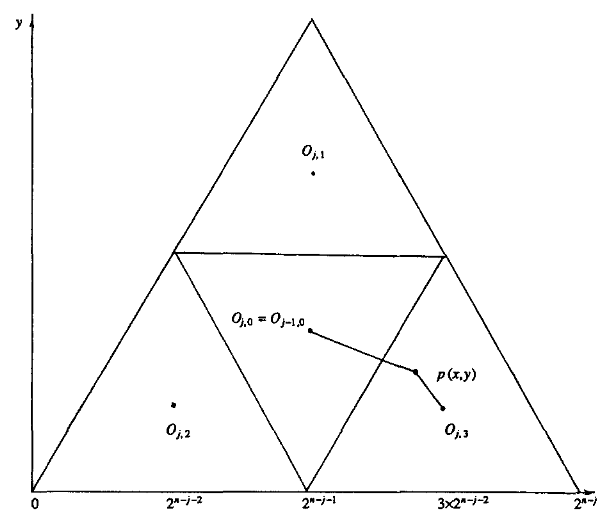


FIG. 4. Centroids of triangular cells and distance to the centroids.

to the centroid of son triangle $k$ ($k = 0, 1, 2$, or 3). This can be seen by drawing three bisectors of the parent triangle and connecting $p(x, y)$ to the centroids of four son triangles as shown in Fig. 4.

We start the recursive approximation procedures from level 0. To simplify the calculation, the distance squared is used instead of distance.

(1) Calculate the relative distance

$$\Delta x_0 = x - 2^{n-1}, \quad \Delta y_0 = y - \frac{2^{n-1}}{\sqrt{3}} \tag{3-1}$$

between $(x, y)$ and the centroid of the original triangle.

(2) Calculate

$$D(1, l) = (\Delta x_0 - \Delta X_{1,l})^2 + (\Delta y_0 - \Delta Y_{1,l})^2 \tag{3-2a}$$

for $l = 0, 1, 2, 3$, where $\Delta X_{1,l}$ and $\Delta Y_{1,l}$ are the relative distance from the centroids of the first-level triangles to the centroid of the 0-level triangle as expressed in (2-4a) to (2-4d) and $\alpha = 1$ for $j = 1$ (level 1). If

$$\min[D(1, 0), D(1, 1), D(1, 2), D(1, 3)] = D(1, k_1), \tag{3-2b}$$

then $p(x, y)$ is in the triangle $k_1$ of the first level triangles and we set

$$a_1 = k_1 \tag{3-2c}$$

and

$$\Delta x_1 = \Delta x_0 - \Delta X_{1,k_1}, \quad \Delta y_1 = \Delta y_0 - \Delta Y_{1,k_1}. \tag{3-2d}$$

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The recursive approximation algorithm for conversion of Cartesian coordinates to triangle address is as follows:

input $x, y, k$;
output $a_1, a_2, a_3, \ldots, a_n$;
**begin** {main}

$$\Delta x_0 = x - 2^n; \qquad \Delta y_0 = y - \frac{2^{n-1}}{\sqrt{3}}$$

$j = 0$;

**repeat**
**for** $j = 1$ **to** $k$
  **for** $l = 1$ **to** 3

$$NZ_j = a_{00} + \sum_{i=1}^{j-1} [a_j = 0]$$

$$D(j, l) = (\Delta x_{j-1} - \Delta X_{j,l})^2 + (\Delta y_{j-1} - \Delta Y_{j,l})^2$$

  **if**

$$D(j, k_j) = \min[D(j, 0), D(j, 1), D(j, 2), D(j, 3)]$$

  **then**

$a_j = k_j$
$j = j + 1$

**continue**
**end**

EXAMPLE 3. Given the Cartesian triangle address $(x, y) = (23 \times 2^{n-5}, (11 \times 2^{n-5})/\sqrt{3})$ find the triangular tiling address $a_1, a_2, a_3, a_4$.

(1) For $j = 0$, we have $\alpha = +1$:

$$\Delta x_0 = (23 - 16)2^{n-5} = 7 \times 2^{n-5}, \qquad \Delta y_0 = \frac{(11 - 16)2^{n-5}}{\sqrt{3}} = -5 \times 2^{n-5}.$$

(2) For $j = 1$, we have $\alpha = +1$:

$$\Delta x_{1,0} = (7 - 0) \times 2^{n-5} = 7 \times 2^{n-5}, \qquad \Delta y_{1,0} = \frac{-(5 - 0) \times 2^{n-5}}{\sqrt{3}} = \frac{-5 \times 2^{n-5}}{\sqrt{3}}$$

$D(1, 0) = 57.33333$:

$$\Delta x_{1,1} = (7 - 0) \times 2^{n-5} = 7 \times 2^{n-5}, \qquad \Delta y_{1,1} = \frac{-(5 + 16) \times 2^{n-5}}{\sqrt{3}} = \frac{-21 \times 2^{n-5}}{\sqrt{3}}$$

$D(1, 1) = 196$:

$$\Delta x_{1,2} = (7 + 8) \times 2^{n-5} = 15 \times 2^{n-5}, \quad \Delta y_{1,2} = \frac{-(5 - 8) \times 2^{n-5}}{\sqrt{3}} = \frac{3 \times 2^{n-5}}{\sqrt{3}}$$

$D(1, 2) = 228$:

$$\Delta x_{1,3} = (7 - 8) \times 2^{n-5} = -2^{n-5}, \qquad \Delta y_{1,3} = \frac{-(5 - 8) \times 2^{n-5}}{\sqrt{3}} = \frac{3 \times 2^{n-5}}{\sqrt{3}}$$

$D(1, 3) = 4$:

$$\min D(1, l) = D(1, 3) = 4.$$

We have

$$a_1 = 3: \qquad \Delta x_1 = -2^{n-5}, \qquad \Delta y_1 = \frac{3 \times 2^{n-5}}{\sqrt{3}}.$$

(3) For $j = 2$, we have $\alpha = +1$ since $a_1 \neq 0$:

$$\Delta x_{2,0} = (-1 - 0) \times 2^{n-5} = -2^{n-5}, \qquad \Delta y_{2,0} = \frac{(3 - 0) \times 2^{n-5}}{\sqrt{3}} = \frac{3 \times 2^{n-5}}{\sqrt{3}}.$$

$D(2, 0) = 4$:

$$\Delta x_{2,1} = (-1 - 0) \times 2^{n-5} = -2^{n-5}, \qquad \Delta y_{2,1} = \frac{(3 - 8) \times 2^{n-5}}{\sqrt{3}} = \frac{-5 \times 2^{n-5}}{\sqrt{3}}$$

$D(2, 1) = 9.333333$:

$$\Delta x_{2,2} = (-1 + 4) \times 2^{n-5} = 3 \times 2^{n-5}, \quad \Delta y_{2,2} = \frac{(3 + 4) \times 2^{n-5}}{\sqrt{3}} = \frac{7 \times 2^{n-5}}{\sqrt{3}}$$

$D(2, 2) = 25.333333$:

$$\Delta x_{2,3} = (-1 - 4) \times 2^{n-5} = -2^{n-5}, \qquad \Delta y_{2,3} = \frac{(3 + 4) \times 2^{n-5}}{\sqrt{3}} = \frac{7 \times 2^{n-5}}{\sqrt{3}}$$

$D(2, 3) = 41.333333$:

$$\min D(2, l) = D(2, 0) = 4.$$

We have

$$a_2 = 0: \qquad \Delta x_2 = -2^{n-5}, \qquad \Delta y_2 = \frac{3 \times 2^{n-5}}{\sqrt{3}}.$$

(4) For $j = 3$, we have $\alpha = -1$ since $\Sigma_{i=1}^{2}(a_i = 0) = 1$:

$$\Delta x_{3,0} = (-1 - 0) \times 2^{n-5} = -2^{n-5}, \qquad \Delta y_{3,0} = \frac{(3-0) \times 2^{n-5}}{\sqrt{3}} = \frac{3 \times 2^{n-5}}{\sqrt{3}}$$

$D(3, 0) = 4$:

$$\Delta x_{3,1} = (-1 - 0) \times 2^{n-5} = -2^{n-5}, \qquad \Delta y_{3,1} = \frac{(3+4) \times 2^{n-5}}{\sqrt{3}} = \frac{7 \times 2^{n-5}}{\sqrt{3}}$$

$D(3, 1) = 17.333333$:

$$\Delta x_{3,2} = (-1 + 2) \times 2^{n-5} = 2^{n-5}, \qquad \Delta y_{3,2} = \frac{(3-2) \times 2^{n-5}}{\sqrt{3}} = \frac{2^{n-5}}{\sqrt{3}}$$

$D(3, 2) = 1.333333$:

$$\Delta x_{3,3} = (-1 - 2) \times 2^{n-5} = -3 \times 2^{n-5}, \quad \Delta y_{3,3} = \frac{(3-2) \times 2^{n-5}}{\sqrt{3}} = \frac{2^{n-5}}{\sqrt{3}}$$

$D(3, 3) = 4.333333$:

$$\min D(3, l) = D(3, 2) = 1.333333.$$

We have

$$a_3 = 2: \qquad \Delta x_3 = 2^{n-5}, \qquad \Delta y_3 = \frac{2^{n-5}}{\sqrt{3}}.$$

(5) For $j = 4$, we have $\alpha = -1$ since $\Sigma_{i=1}^{3}(a_i = 0) = 1$:

$$\Delta x_{4,0} = (1 - 0) \times 2^{n-5} = 2^{n-5}, \qquad \Delta y_{4,0} = \frac{(1-0) \times 2^{n-5}}{\sqrt{3}} = \frac{2^{n-5}}{\sqrt{3}}.$$

$D(4, 0) = 1.333333$:

$$\Delta x_{4,1} = (1 - 0) \times 2^{n-5} = 2^{n-5}, \qquad \Delta y_{4,1} = \frac{(1+2) \times 2^{n-5}}{\sqrt{3}} = \frac{3 \times 2^{n-5}}{\sqrt{3}}$$

$D(4, 1) = 4$:

$$\Delta x_{4,2} = (1 + 1) \times 2^{n-5} = 2 \times 2^{n-5}, \qquad \Delta y_{4,2} = \frac{(1-1) \times 2^{n-5}}{\sqrt{3}} = 0$$

$D(4, 2) = 4$:

$$\Delta x_{4,3} = (1 - 1) \times 2^{n-5} = 0, \qquad \Delta y_{4,3} = \frac{(1-1) \times 2^{n-5}}{\sqrt{3}} = 0$$

$D(4, 3) = 0$:

$$\min D(4, l) = D(4, 3) = 0.$$

We have

$$a_4 = 3: \qquad \Delta x_4 = 0, \qquad \Delta y_4 = 0.$$

Therefore, the triangular tiling address is

$$A = a_1\, a_2\, a_3\, a_4 = 3023.$$

## 4. THE AREA OF DECOMPOSED TRIANGLES

The process we have described for creating an HSDS for the globe does not satisfy properties (1) and (2) precisely; triangles at level $k > 0$ are not equal in area and have varying shapes, although we believe that our scheme represents a reasonable compromise between these conflicting objectives. In this section we examine the areas of triangles explicitly. We assume that the earth is spherical, although the results should generalize easily to the more accurate ellipsoid of revolution.

The earth surface area $A$ between latitude $\phi_1$ and $\phi_2$ covered by a level-0 triangle is

$$A = \frac{\pi R^2}{2} (\sin \phi_2 - \sin \phi_1). \tag{4-1}$$

At level $n$, the total number of triangles in the belt between $\phi$ and $\phi + \pi/2^{n+1}$ is

$$N_\phi = 2 \times \frac{2^{n+1}}{\pi}\left(\frac{\pi}{2} - \phi\right) - 1 = \frac{2^{n+2}}{\pi}\left(\frac{\pi}{2} - \phi\right) - 1$$

and the Earth surface area of a triangle at level $n$ is

$$\Delta A_\phi = \frac{A}{N_\phi} = \frac{\pi^2 R^2}{2^{n+3}} \frac{\sin(\phi + 2^{-n-1}\pi) - \sin \phi}{\pi/2 - \phi - 2^{-n-2}\pi}. \tag{4-2}$$

When $2^{n+1} \gg 1$, we have

$$\sin(\phi + 2^{-n-1}\pi) - \sin \phi \approx \frac{\pi}{2^{n+1}} \cos \phi$$

and

$$\frac{\pi}{2} - \phi - 2^{-n-2}\pi \approx \frac{\pi}{2} - \phi.$$

Expression (4-2) can be written as

$$\Delta A_\phi \approx k \frac{\cos \phi}{\pi/2 - \phi} = k \frac{\sin \chi}{\chi} = k \operatorname{sinc} \chi, \tag{4-3}$$

where

$$k = \frac{\pi^3 R^2}{2^{2n+4}}, \quad \chi = \frac{\pi}{2} - \phi.$$

From expression (4-3), it is interesting to note that the area covered by a high-level decomposed triangle varies with the sinc function. In the range from $\chi = 0$ to $\chi = \pi/2$, $\operatorname{sinc} \chi$ is a monotonically decreasing function of $\chi$, or $\Delta A_\phi$ is an increasing function of $\phi$ for $\phi = 0$ to $\phi = \pi/2$. For $\phi = 0$ and $\phi = \pi/2$, we have

$$\Delta A_0 = \frac{2k}{\pi}, \quad \Delta A_{\pi/2} = k$$

and

$$\frac{\Delta A_{\pi/2}}{\Delta A_0} = \frac{\pi}{2}. \tag{4-4}$$

That is, for high-level decomposed triangles, the corresponding area increases $\pi/2 = 1.5708$ times when latitude changes from 0 to $\pi/2$. Only the triangles along a given latitude have the same area and the area changes with latitude according to the sinc function as shown in Fig. 5.

## 5. ALGORITHM FOR FINDING NEIGHBORS OF TRIANGLES

Algorithms for finding neighbors in hierarchical recursive schemes such as the quadtree have been described by Samet [4, 11, 12] and others. The problem of finding neighbors in our scheme is more difficult because of the special conditions at octahedron edges.

It is often necessary to find the *three directly connected neighbors* of a given triangle with address

$$A = a_1, a_2, a_3, \ldots, a_k.$$

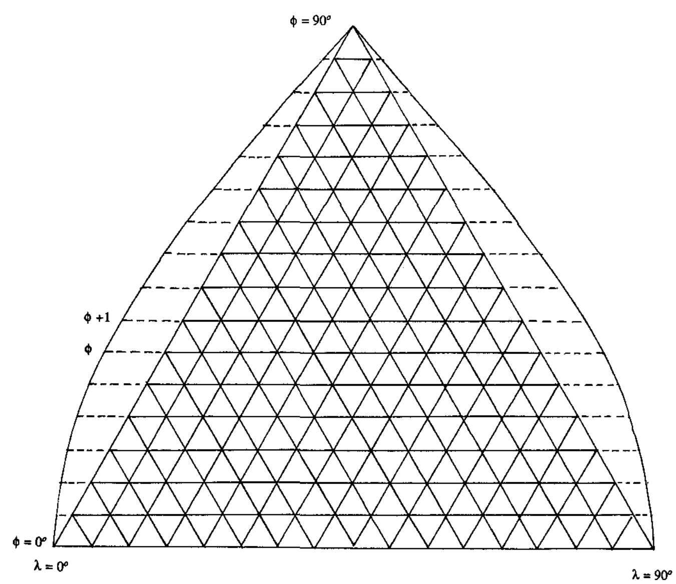


FIG. 5. Distribution of the area of triangular cells at different latitudes (sinc function).

We denote the three neighbors as *top, left,* and *right* neighbors. The direction of the neighbors depends on whether the triangle $A$ is upward or downward as follows:

| | Top | Left | Right |
|---|---|---|---|
| Upward (NZ even) | S | NW | NE |
| Downward (NZ odd) | N | SW | SE |

We use the codes

$$T = t_1, t_2, t_3, \ldots, t_k,$$

$$L = l_1, l_2, l_3, \ldots, l_k,$$

$$R = r_1, r_2, r_3, \ldots, r_k,$$

to represent the addresses of the top, left, and right neighbors, respectively. Both the triangle and its neighbor are inside a triangle of the $j$th level if $a_1, a_2, \ldots, a_{j-1}$ does not change. The problem is to determine the level of triangle within which a neighbor of a given triangle is contained and to change the code of $a_j, a_{j+1}, \ldots, a_k$ for the top, left, and right neighbors of a given triangle separately.

Recall that the triangles are ordered as follows:

(1) the center, top, left and right triangles within a triangle are ordered 0, 1, 2, and 3 respectively;

(2) the triangle with $a_j = 1$ is a reflection of the triangle with $a_j = 0$;

(3) the triangles with $a_j = 2$ and $a_j = 3$ are the left-down (or left-up) and right-down (or right-up) translation of the triangle with $a_j = 1$, respectively.

The neighbor addresses can be searched using the following conversion table:

| $a_i$ | Top | Left | Right |
|---|---|---|---|
| 0 | 1# | 2# | 3# |
| 1 | 0# | 3 | 2 |
| 2 | 2 | 1 | 0# |
| 3 | 3 | 0# | 1 |

for $a_i$ ($i$ from $k$ to $j$), where # is the search terminate symbol. It can be implemented as follows.

(1) To find the top neighbor, starting from $i = k$, $a_i$ changes to $t_i$,

$$t_i = 2 \quad \text{if} \quad a_i = 2$$

$$t_i = 3 \quad \text{if} \quad a_i = 3,$$

and process $a_{i-1}$ for $t_{i-1}$. On the other hand if $a_i \neq 2$ and $a_i \neq 3$, then

$$t_i = 1 \quad \text{if} \quad a_i = 0$$

$$t_i = 0 \quad \text{if} \quad a_i = 1$$

and the search finishes. We define this $i$ as $j$ and set

$$t_k = a_k$$

for $i = 1$ to $i = j - 1$.

(2) For the left neighbor, starting from $i = k$, $a_i$ changes to $l_i$,

$$l_i = 3 \quad \text{if} \quad a_i = 1$$

$$l_i = 1 \quad \text{if} \quad a_i = 2,$$

and process $a_{i-1}$ for $l_{i-1}$. On the other hand if $a_i \neq 1$ and $a_i \neq 3$, then

$$l_i = 2 \quad \text{if} \quad a_i = 0$$

$$l_i = 0 \quad \text{if} \quad a_i = 3$$

and the search finishes. We define this $i$ as $j$ and set

$$l_k = a_k$$

for $i = 1$ to $i = j - 1$.

(3) For the right neighbor, starting from $i = k$, $a_i$ changes to $r_i$,

$$r_i = 2 \quad \text{if} \quad a_i = 1$$

$$r_i = 1 \quad \text{if} \quad a_i = 3,$$

and process $a_{i-1}$ for $r_{i-1}$. On the other hand if $a_i \neq 1$ and $a_i \neq 3$, then

$$r_i = 3 \quad \text{if} \quad a_i = 0$$

$$r_i = 0 \quad \text{if} \quad a_i = 2$$

and the search finishes. We define this $i$ as $j$ and set

$$r_k = a_k$$

for $i = 1$ to $i = j - 1$.

The algorithm for finding neighbor addresses described above is easy to implement as only $k - j + 1$ quaternary digits have to be determined by simple criteria and the other $j - 1$ digits are only a copy of the corresponding digits in the given triangle. The average number of quaternary digits that need to be changed to find a neighbor can be determined as follows:

(1) The probability of changing only the last ($k$th) digit is $\frac{1}{2}$;

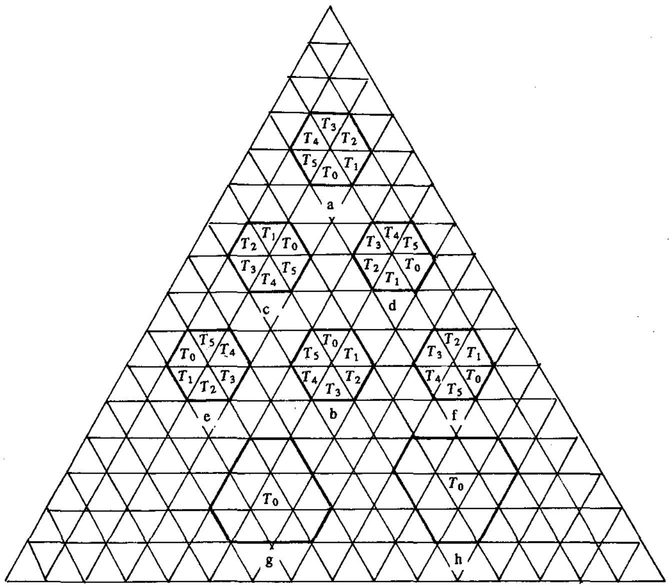

FIG. 6. Hexagon including a given triangle, and 12 neighbors of a given triangle: (a) top hexagon with given triangle upward; (b) top hexagon with given triangle downward; (c) left hexagon with given triangle upward; (d) left hexagon with given triangle downward; (e) right hexagon with given triangle upward; (f) right hexagon with given triangle downward; (g) 12 neighbors with given triangle upward; (h) 12 neighbors with given triangle downward.

(2) The probability of changing $j$ quaternary digits is $2^{-j}$.

Therefore, the *average number of steps of calculation* is

$$S_{av} = \sum_{j=1}^{k} \frac{j}{2^j} + \frac{k}{2^k} = 2 - \frac{2}{2^k}.$$

$S_{av}$ is less than two quaternary or four binary digits.

The neighbor finding algorithm can be used for searching hexagons with a given triangle included or to find the 12 neighbors of a given triangle (Fig. 6). Let a given triangle be denoted by $T_0$ and the other five triangles in a hexagon are $T_1$, $T_2$, $T_3$, $T_4$, and $T_5$, respectively. They can be found as follows whether the given triangle is upward or downward.

For a hexagon with the given triangle as a top triangle,

$$T_1 = R(T_0), \quad T_2 = T(T_1), \quad T_3 = L(T_2),$$
$$T_4 = L(T_3) \quad T_5 = T(T_4).$$

For a hexagon with the given triangle as a left triangle,

$$T_1 = L(T_0), \quad L_2 = L(T_1), \quad T_3 = T(T_2),$$
$$T_4 = R(T_3), \quad T_5 = R(T_4).$$

For a hexagon with the given triangle as a right triangle,

$$T_1 = T(T_0), \quad T_2 = R(T_1), \quad T_3 = R(T_2),$$
$$T_4 = T(T_3), \quad T_5 = L(T_4).$$

where $T_j = T(T_i)$, $T_j = L(T_i)$ and $T_j = R(T_i)$, which imply that the triangle $T_j$ is the top, left, and right neighbor of triangle $T_i$, respectively. The first 4 triangles in the 3 hexagons above are the 12 neighbors of the given triangle as shown in Fig. 6.

Another application of the neighbor finding algorithm is that a chain code with a series of codes $T$(top), $L$(left), and $R$(right) can be used to describe lines or borders of areas.

## 6. AVERAGE DATA FILE STORAGE DISTANCE

One of the important indices in data file structures for large geographical information systems is the *Average Data File Storage Distance* [5, 13]. This is defined as the average absolute difference between the addresses of neighboring cells or tiles; in our case, each triangle is assumed to have three neighbors. Goodchild and Grandfield [13] used the index in a study of the data compression achieved by different ordering of a lattice, whereas Goodchild [14] argued its usefulness in predicting the time required to access database partitions in very large spatial archives. In this analysis we are concerned only with subdivisions of the level-0 triangles and ignore the differences that occur across edges of the octahedron.

The average data file storage distance is the sum of absolute differences between adjacent triangular cells $D_{total}$ divided by the number of edges $OC_{total}$, or

$$D_{av} = \frac{D_{total}}{OC_{total}}.$$

The total differences for triangular cells can be represented as

$$D_j = 4 \times D_{j+1} + \Delta D_j$$

and

$$OC_j = 4 \times OC_{j+1} + \Delta OC_j,$$

where $4 \times D_{j+1}$ and $4 \times OC_{j+1}$ are the total differences and number of edges of triangular cells for level $j + 1$ triangles and $\Delta D_j$ and $\Delta OC_j$ are the distances and edges added at the $j$th level (see Fig. 1b).

$$\Delta D_j = 6 \times 2^{k-j} \times 4^{k-j} = 6 \times 2^{3(k-j)}$$

is the sum of edge cell values of triangles 1, 2, and 3 minus the sum of edge cell values of triangle 0 at the $j$th level, and

$$\Delta OC_j = 3 \times 2^{k-j}$$

is the total number of edges added at the $j$th level. Therefore

$$D_j = 6 \times \sum_{i=2(k-j)}^{3(k-j)} 2^i = 6 \times 2^{2(k-j)} \times \sum_{i=0}^{k-j} 2^i$$

and

$$OC_j = 3 \times \sum_{i=k-1}^{2(k-1)} 2^i = 3 \times 2^{k-1}(2^k - 1).$$

The average distance at the $j$th level is

$$D_{\text{av},j} = \frac{D_j}{OC_j} = 2^{k-j+1}.$$

We have $j = 1$ when the 0-level triangle is decomposed to $k$th level. In this case

$$D_{\text{av}} = 2^k.$$

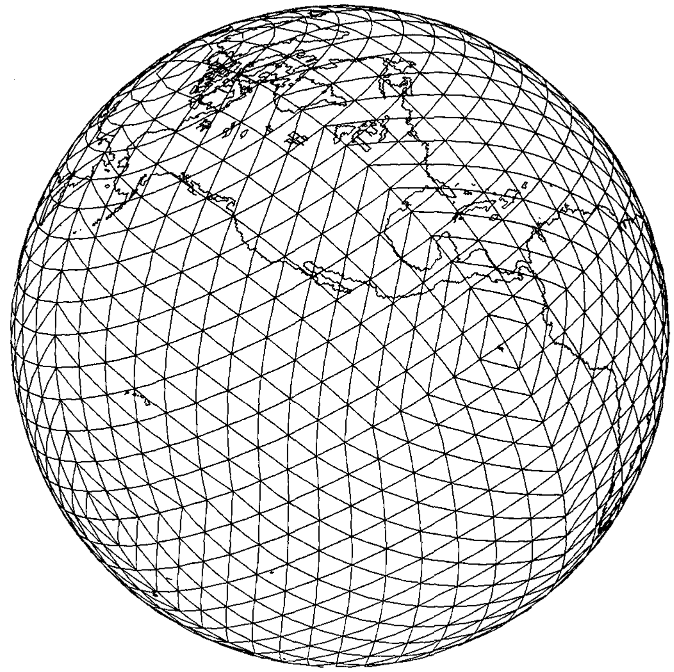

FIG. 7. The level-4 system of triangles viewed orthographically from over the Pacific Ocean at approximately 110°W 15°E.

## 7. CONCLUSIONS

The hierarchical data structure that we have described in this paper satisfies one of our original requirements in full, by preserving the relationships between neighboring cells. The distortions of area inherent in the structure, and described by the sinc function, range up to a factor of 1.57 at the poles. Triangles become increasingly equilateral toward the center of each level-0 triangle at higher levels of subdivision, but the triangles adjacent to each level-0 vertex always contain one right angle. Our requirements of equal area and equal shape are thus satisfied only approximately. Figure 7 shows the level-4 system of triangles viewed orthographically from over the Pacific Ocean at approximately 110°W 15°E.

In this structure every object on the Earth's surface can be indexed by the address of the smallest enclosing triangle. The length of the address is then a direct index of the object's size. To find the smallest enclosing triangle of a polygon, we simply determine the triangle address of one of its vertices to some arbitrary but high level $k$ and then identify a largest value $j \leq k$ such that all other vertices share the same quaternary digits 1 through $j$. For example, the lower 48 states of the United States, which span two level 0-triangles, have a null address, while the block formed by 3rd and 4th Streets, Broadway, and Fulton in the City of Troy, New York, has the address 30223022113013 (level 13). The approximate edge length of a level-16 triangle is 150 m, or the rough dimensions of a city block, according to Table 2. However, while the Broadway and Fulton faces of the block are both wholly within level-16 triangles, the smallest triangle enclosing the entire block is at level 13. We have computed the triangle address of the residence of Professor Waldo Tobler in Santa Barbara. Using the 1983 North America Datum (NAD 83), the latitude/longitude location at 119°48′26″W longitude, 34°26′41″N latitude translates to a level-18 address of 210203230223201331l, with an accuracy of approximately 40 m. The change of datum from NAD 27 produced a 3″ change in longitude, which is equivalent to a change of digits 16–18. The reverse operation of conversion from triangle address to latitude/longitude can be implemented very efficiently in bitwise form.

Length of address can also be used as a measure of uncertainty of position, by identifying the smallest triangle that encloses the union of the object's possible positions. For example, the accuracy currently provided by the Global Positioning System (GPS) is about 20 m. The corresponding length of address for any point on the earth's surface is 19 quaternary digits or 38 bits, any further precision being spurious. For comparison, to achieve 20 m precision in latitude/longitude coordinates, it is necessary to specify location to the nearest second, which requires seven decimal digits plus sign for longitude and six digits plus sign for latitude.

The ideal workstation for global systems modeling would allow the user to browse freely through data distributed over the surface of the globe. With datasets based on rectangular subdivision of a cylindrical projection it is relatively easy to browse in the equatorial region, but difficult near the poles because of high levels of distortion and interruption at the pole itself. Similar problems occur using rectangular subdivision of any other standard projection. For example, the orthographic projection gives a view of the globe as it would appear from space. However, it would be time consuming to recompute and redisplay the projection for every change of viewpoint.

Recent developments in 3D graphics display technology may make browsing on the globe much more practical. Instead of projecting to a plane, a solid is represented digitally by a polyhedron with triangular faces and displayed in perspective directly from a display list of triangles. The graphical rendering (color or texture) of each triangle can be controlled directly from its attributes. Workstations that can display polyhedra of 10,000 triangles in 1 s are currently available for less than $20,000, and we can expect orders of magnitude improvement in these specifications in the near future. Thus we are able with current technology to create a browse of a global dataset at level 6 (approximately 1° resolution). For spatial variables such as land/water, subdivision can be much higher in some areas because of the relative homogeneity of continents and oceans. Thus the developing technology of 3D display based on polyhedra with triangular faces gives a powerful argument for triangle-based tessellation over more conventional methods.

The results presented in this paper suggest several potentially fruitful areas for further work. We have thus far ignored the nonspherical nature of the earth in calculating triangles areas. Associated algorithms, particularly for building the triangle data structure from vector data, such as the world's coastlines, have been developed and will be described in subsequent papers. In the longer term, we plan to develop a prototype workstation for global data based on the triangular structure and triangle display lists.

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