# Thirty Five Years of Computer Cartograms

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**Abstract**

The notion of a cartogram is reviewed. Then, based on a presentation from the 1960's, a direct and simple introduction is given to the design of a computer algorithm for the construction of contiguous value-by-area cartograms. As an example a table of latitude/longitude to rectangular plane coordinates is included for a cartogram of the United States, along with Tissot's measures for this map projection. This is followed a short review of the subsequent history of the subject and includes citation of algorithms proposed by others. The most common use of cartograms is solely for the display and emphasis of a geographic distribution, as a contrast to the usual geographic map. A second use is in analysis, as a nomograph or problem solving device similar in use to Mercator's projection, or in the transform-solve-invert paradigm. Recent innovations by computer scientists modify the objective and suggest variation similar to Airy's (1861) 'balance of errors' idea for map projections.

**Keywords**: Anamorphoses, cartograms, map distortion, map projections, quasiconformal.

The first reference to the term "cartogram" that I have found is to Minard, as follows: "In 1851 Minard published a series of maps called "'cartogrammes a foyer diagraphiques' or maps with diagrams…." (Friis 1974, page 133). The term is not listed in Robinson's (1967) paper on Minard, nor in his book on the history of thematic cartography (Robinson 1982). He only briefly mentions the related choropleth maps by name, although he gives illustrations of several such maps from the mid 1800s. He also gives a map by Minard from 1850 in which deliberate distortion occurs in order to make room for the symbols on an illustration depicting flow (also see Robinson 1967, pages 101-102). The first entry in Wallis and Robinson's (1987) survey on *Cartographical Innovations* is the term 'cartogram' but they do not give the derivation. Wright uses the term in his introduction to Paullin's historical atlas of the United States (1932, page xiv) and comments "A cartogram is a cartographic outline upon which are drawn statistical symbols that do not conform closely to the actual distribution of the phenomena represented". He is of course referring to what is now called a statistical map. Kretschmer, et al. (1986, vol. 1, page 396), in their history of cartography under the heading 'Kartogramm' mean a choropleth or statistical map, but also refer to a 'verzerrte' (distorted) map by Wiechel from 1903. There is also a reference to map of Germany from 1903 in which statistics are shown on a schematic map (see Mayet 1905). In this country The Washington Post on Sunday, November 3rd, 1929, printed a map of the United States with state areas equal to population and taxation, accompanied by a

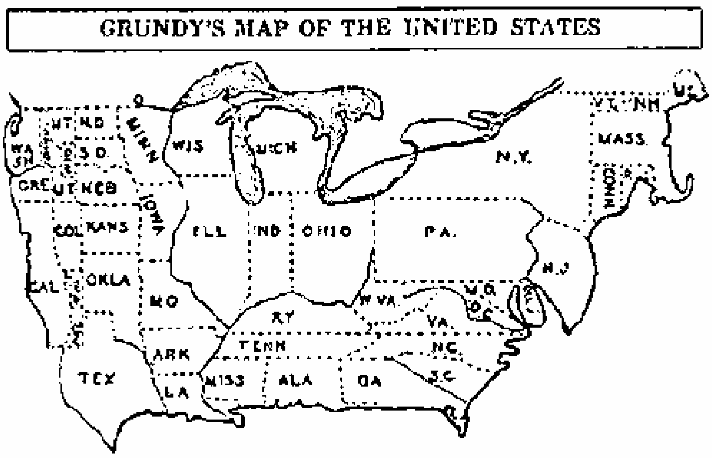



__figure 1:

"Joseph R. Grundy, Pennsylvania manufacturer, suggested in the Senate lobby committee that the present equal power of States in voting on tariff bills is unfair because of differences in voting strength. Here's a map of the United States showing the size of each State on the basis of population and Federal Taxes". From the Washington Post November 3, 1929.

proposal to the Congress to modify the allocation of tariffs (Figure 1). This map would now be called a contiguous cartogram and shows Grundy's home state of Pennsylvania enlarged as are the industrial states of Illinois, Michigan, New York, New Jersey, and Ohio.

The term ‘cartogram’ was used repeatedly by Funkhouser in his history of graphical methods (1937). But by cartogram he means what we now call choropleth maps. Raisz, in the first American cartography textbook *General Cartography* states that the term “cartogram is subject to many interpretations and definitions”. He continues “Some authors, especially in Europe, call every statistical map a cartogram, because it shows the pattern of distribution of a single element.”, in contradistinction to a topographic map which combines many elements. He then has a section, as follows (op. cit., page 257):

> “**Value-Area Cartograms**. In these cartograms a region, country, or continent is subdivided into small regions, each of which is represented by a rectangle. This rectangle is proportionate in area to the value which it represents in certain statistical distributions. The regions are grouped in approximately the same positions as they are on the map.”

In this presentation, as in his earlier papers from 1934 and 1936, Raisz only refers to rectangular cartogramatic diagrams. Raisz (1934, page 292) also asserts that “…the statistical cartogram is not a map.” In his textbook *Principles of Cartography* (1962, 215) Raisz states that “A *cartogram* may be defined as a *diagrammatic* map.” This suggests that the term ‘cartogram’ is a contraction of the two words, given the flexibility of our languages in constructing such combinations.

Interestingly Funkhouser’s paper actually presents a map that now might be called an area cartogram. This is a map of the countries of Europe in which each country is represented by a square whose size is proportional to the area of the country, and with countries in their approximately correct position and adjacency (Figure 2). Could this be called an equal area map? Or is it an equal area cartogram? The map is dated 1870 and was used by the French educator


Figure 2:

Levasseur's cartogram of Europe showing countries in their correct size. (After Funkhouser)

Levasseur in one of his textbooks. Raisz (1934, 1936) also has such ‘equal-land-area’ rectangular cartograms of the United States, as well as some displaying other phenomena. Looking at these examples it is clear that one should distinguish between cartograms that rigorously maintain the correct adjacency table (O'Sullivan and Unwin 2002; 40, 155) and those that do not.

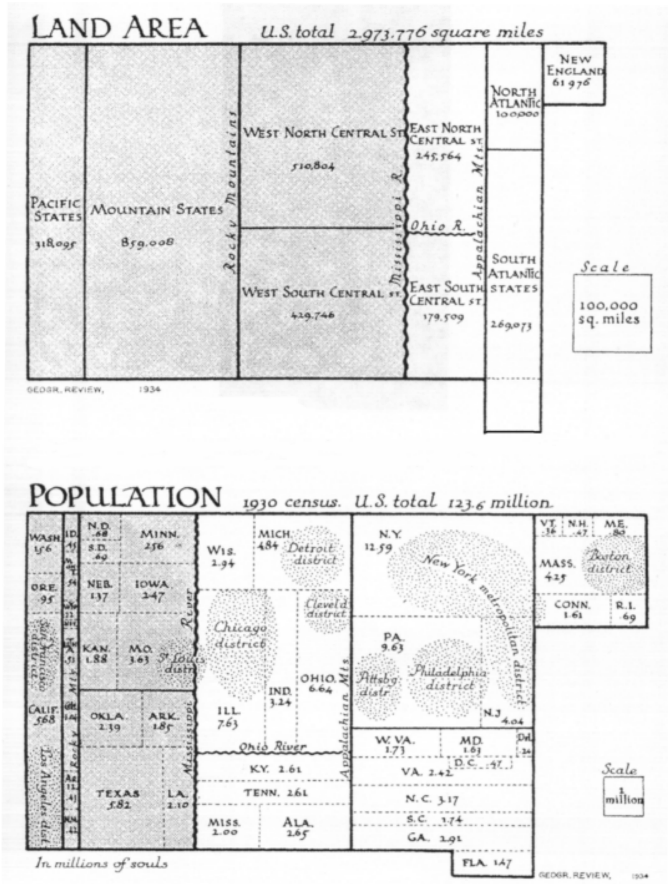



Figure 3:

Two of Raisz's (1934) rectangular cartograms. The top diagram is a statistical cartogram showing the land surface (km$^2$) by Census divisions. Below this is a "Rectangular statistical cartogram with rectangles representing geographical divisions of the Census and states proportionate in size to their population". Comparing the differential position of the Appalachian Mountains and the Mississippi River on the two diagrams dramatically illustrates the differences. Additional cartograms in the paper include representations of "National Wealth", "Value Added by Manufacture", "Farm Products", "Mine & Quarry Products", "Crude Oil", "Sand & gravel", and "Natural Gas", all scaled in 1920 dollars. From Raisz, E., 1934, “The Rectangular Statistical Cartogram”, *Geographical Review*, 24(2): 293. Used by permission of the American Geographical Society.

Diagrams such as those of Levasseur and Raisz have adjacency tables that have both too many and too few correct adjacencies. In those that do display the correct topological adjacency it is worth noting whether or not the maps are continuous, with continuous partial derivatives, or only piecewise continuous.

A discussion of ‘value-by-area’ cartograms can now be found in several contemporary cartographic textbooks, e.g., Dent (1999, pages 207-219), Slocum (1999, pages 181-184). In Canters’ recent book on map projections (Canters 2002, pages 157-167) they are included as ‘variable scale maps’. The French use the term ‘anamorphose’ (for a derivation of this term see Hankins 1999), in German speaking countries ‘verzerrte Karte’ is most prevalent, and the Soviets have used the word ‘varivalent’ maps. These terms are a bit misleading since the scale on a geographic map is never a single constant, though the meaning here is clear.

I became interested in the subject in 1959 and my doctoral thesis on *Map Transformations of Geographic Space* (Tobler 1961) was devoted to this topic. There are now two generally accepted types of cartograms. One form stretches space according to some metric different from kilometers or miles, such as cost or time. These are not considered in this review, although they constituted the bulk of the above-cited thesis. I also do not consider the piecewise continuous (non-contiguous) cartograms described by Olson (1976). The concern here is with the type that stretches space continuously according to some distribution on a portion of the earth's surface. These are generally referred to as area (or areal) cartograms, or (following Raisz) as value-by-area maps. The single chapter of my dissertation that was devoted to this topic was published in the *Geographical Review* (Tobler 1963), but with a mathematical appendix that was not in the thesis. When I arrived at the University of Michigan in Ann Arbor I began to develop computer programs that could compute area cartograms. The easiest way for me to introduce this subject is to reproduce a lecture given in the 1960's to Howard Fisher's computer graphics group at Harvard University. The following is an transcription of the original overhead viewgraphs from that lecture. It remains a simple and clear definition of the problem and of one solution.

**The Harvard Presentation**

A value-by-area cartogram is a map projection that converts a measure of a non-negative distribution on the earth to an area on a map. Consider first a distribution h (u, v) on a plane (Figure 4):

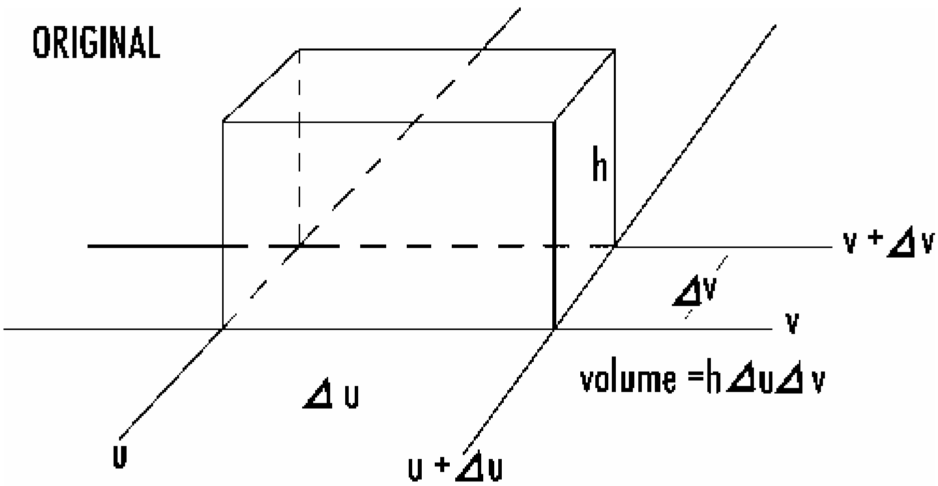



Figure 4:

A representation of the to-be-preserved density on a plane.

Next consider an area on the map (Figure 5):

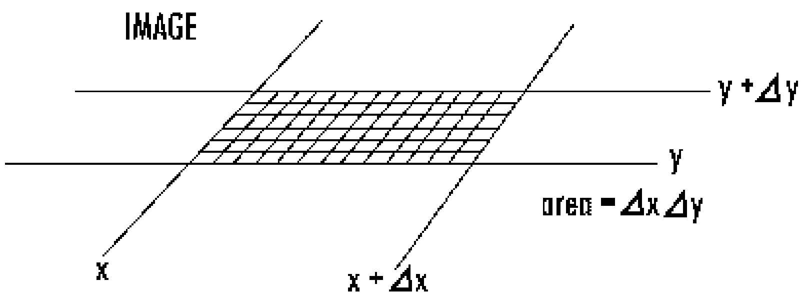



Figure 5:

The density converted to a plane area.

We want these to be the same. That is, the map image is to equal the original measure: *image area on map equals original volume on surface*, or $\Delta x\ \Delta y = h\ \Delta u\ \Delta v$.

As an aside, observe that in his treatise of 1772 J. H. Lambert defined an equal area map projection in exactly this fashion, setting spherical surface area equal to map surface area, of course with the cosine of the latitude included to account for a spherical earth.

Now replace the Δ's by d's, i.e. dx dy = h du dv. This is for one unit area but it can be rewritten in integral form to cover the entire domain as

$$\int\int dx\, dy = \int\int h\, du\, dv$$

To solve this system we can insert a transformation, i.e., divide both sides by dudv to get dxdy/dudv = h (u, v). This can be recognized as the introduction of the Jacobian determinant, as covered in beginning calculus courses. With this substitution the condition equation becomes

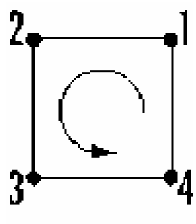
$$J = \partial (x, y)/\partial (u, v) = h (u, v).$$

Written out in full we have the equation

$$\partial x/\partial u\ \partial y/\partial v\ -\ \partial x/\partial v\ \partial y/\partial x = h (u, v). \qquad [1]$$

To apply this mildly non-linear partial differential equation to a sphere it is only necessary to multiply by $R^2 \cos \varphi$ on the right hand side of this equation, substituting longitude (λ) and latitude (φ) for the rectangular coordinates u and v. That a valid value-by-area preserving map can exist follows from the solution properties of this differential equation.

Reverting now to the pictures on the plane, a small rectangle will have nodes identified by cartesian coordinates given in a counterclockwise order (Figure 6).




Fgure 6

quare with vertex identification. Counterclockwise numbering is indicated.

The area A of such a rectangle is given by the determinant formula

$$2A = \begin{vmatrix} X_1 & Y_1 \\ X_2 & Y_2 \end{vmatrix} + \begin{vmatrix} X_2 & Y_2 \\ X_3 & Y_3 \end{vmatrix} + \begin{vmatrix} X_3 & Y_3 \\ X_4 & Y_4 \end{vmatrix} + \begin{vmatrix} X_4 & Y_4 \\ X_1 & Y_1 \end{vmatrix}$$

It is now desired that this area be made equal to the "volume" h (u, v). This can be done by adding increments Δx, Δy to each of the coordinates. The new area can be computed from a determinant formula as above but now using the displaced locations $X_i + \Delta X_i$ , $Y_i + \Delta Y_i$ , i = 1, … , 4 (Figure 7).

Call the new area A' ("a prime"). Now use the condition equation to set the two areas equal to each other, A' = A. Recall that A' is, by design, numerically equal to h (u, v). The equation A' = h = A involves eight unknowns, the $\Delta X_i$ and the $\Delta Y_i$ , i = 1…4 for the rectangle (Figure 8). Now it makes sense to invoke an isotropicity condition, to attempt to retain shapes as nearly as possible. So set all $\Delta X_i$ and $\Delta Y_i$ equal to each other in magnitude, and simply call the resulting value Δ. That is, assume

$$\Delta X_2 = -\Delta X_1 \quad \Delta Y_2 = \Delta Y_1$$
$$\Delta X_3 = -\Delta X_1 \quad \Delta Y_3 = -\Delta Y_1$$
$$\Delta X_4 = \Delta X_1 \quad \Delta Y_4 = -\Delta Y_1$$

and finally that $\Delta = \Delta X_1 = \Delta Y_1$ (Figure 9). This is also the condition that the transformation be, as nearly as possible,

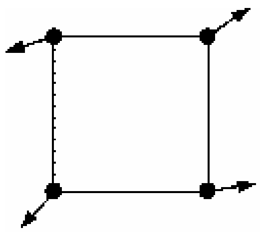


Figure 7

Possible vertex displacements, with no constraints.

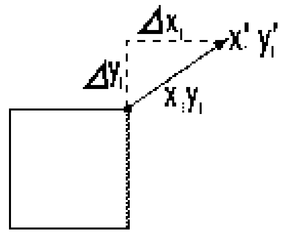



Figure 8

Notation for the displacement of one vertex

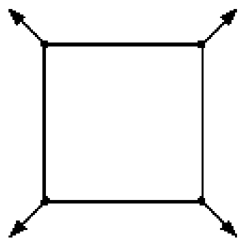


Figure 9:

Uniform displacement of all vertices, yielding a similarity transformation.

conformal, that is, locally shape preserving, and minimizes the Dirichlet integral

$$\int_R (\partial x^2/\partial u + \partial y^2/\partial v + \partial x^2/\partial v + \partial y^2/\partial u)\, du dv,$$

And this renders the transformation unique. Thus A' is just an enlarged (or shrunken) version of A, a similitude. Working out the details yields

$$A' = 4\Delta^2 + \Delta (X_1 - X_2 - X_3 + X_4 + Y_1 + Y_2 - Y_3 - Y_4) + A$$

This quadratic equation is easily solved for the unknown $\Delta$. Once This quantity is found the problem has been solved, but for only one piece of territory. The map area A' is now numerically equal to the numerical value of h (u, v) for that piece.

Many different equal area map projections are possible, and the same holds for value-by-area cartograms. There is one defining equation [eq. 1, above] to be satisfied but this is not sufficient to completely specify a map projection. Two conditions are generally required to determine a map projection, or the differential equation must be given boundary conditions. The choice of a transformed map that looks as nearly as possible like the original was made above and this minimizes the angular distortion, locally preserving shape as nearly as possible. Sen (1976) presents another possible condition. In his example, based on population for the United States, he retains the latitude lines as equally spaced horizontal parallel lines, but the example, while preserving the desired property, is hardly legible. It is also possible to formulate the problem in polar coordinates, as for azimuthal map projections. Three different versions of this possibility are given in Tobler (1961 & 1963, 1973 & 1974, and 1986).

If the regions with which one is dealing are irregular polygons, instead of rectangles, the procedure is exactly the same. One simply translates to the centroid of a polygon and then expands (or shrinks) by the proper amount to get the desired area size, getting a local similitude. In order to do this use the same reasoning as above, expanded to cover general polygons instead of rectangles. The result is that the scale change is the square root of (A'/A) , from which Δ is easily calculated (Figure 10).

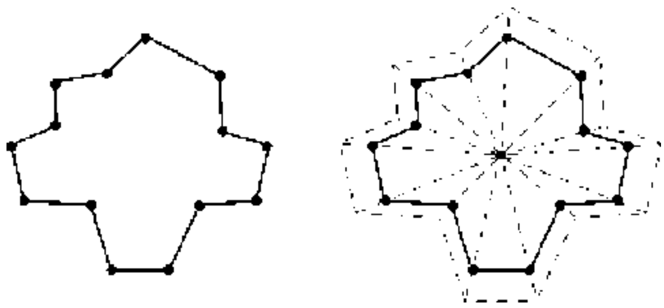


Figure 10:

A polygon and displacements to yield a similarity transformation by expansion from the centroid.

When two rectangles, or two polygons, are attached to each other they will have nodes in common. Then the amounts of displacement calculated independently for a single node associated with more than one area will differ. Suppose one displacement is calculated to be $\Delta_1$ and the other $\Delta_2$. Then take the (vector) average. But this means that neither will result in the desired displacement. In particular, in a set of connected rectangles, or polygons, this problem will occur at almost all nodes. And some nodes will be connected to more than two regions (Figure 11). Again, just average all of the displacements. After all of these displacements have been

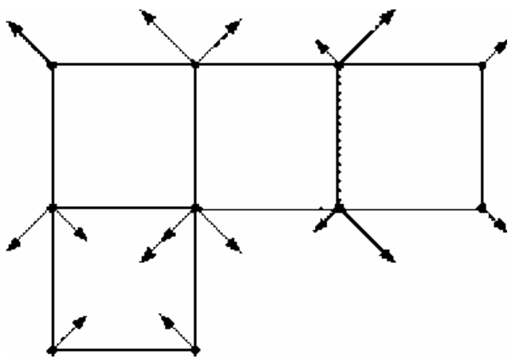


Figure 11:

Illustrating conflicting displacements. One node may need to be displaced to two (or more) different positions. The resolution of this conflict is described in the text..

calculated apply them by adding the increments to every node. Then repeat the process in a convergent iteration. Eventually all regions will converge to their proper, desired size.

There remains one final problem. The displacement at a node can be such that it requires that the node cross over the boundary of some region. This must be prevented. Equally seriously, displacing two nodes may result in the link between them crossing over some other node. And this again must be prevented. Both of these situation are illustrated in the accompanying figure (Figure 12).

Figure 12:

Illustrating a vertex crossing a line, and a line crossing a vertex. Both are violations of the topology.

Under-relaxation - shrinking the displacements to some fraction, say 75%, of the desired values helps avoid, but does not prevent, the problem. The technique used in my computer programs to solve this difficulty was to look at all adjacent nodes, three at a time. This triple makes up a triangle. When a node of the resulting triangle is displaced to cross

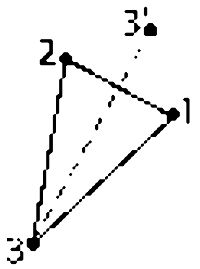



Figure 13:

Adjusting the topology by checking the algebraic sign of the area of a triangle. See text.

over the opposite edge of the triangle, as in Figure 13, then the problem (a node crossing over a line, or a line crossing over a node) has occurred. Turning a triangle inside out in this fashion changes the algebraic sign of the triangular area, from positive to negative. When this happens the displacement must be adjusted. Therefore, if a crossing is detected in the triangle, then the displacement is shrunken until it is no longer is a problem. Thus every node must be checked against every boundary link, and every link must be compared to every node. This can get quite tedious, consuming considerable computer time. This topological checking slows the algorithm down considerably. The next iteration computes a new set of displacements and the desired result is eventually achieved, in a convergent iteration. The topological check also prevents negative areas from occurring. Negative areas are not permitted, since by assumption, h (u, v) is non-negative.

The "error", the discrepancy between the desired result and the result obtained, is measured by $\sum |A' - A| / \sum A$ with A' normalized so that $\sum A' = \sum A$ over all areas. The convergence of the algorithm then follows a typical monotonic exponential decay. My experience on an IBM 709 computer, with data given by latitude and longitude quadrilaterals and using one degree population data (a 25 by 58 lattice of 1450 cells) for the continental United States, was that the program required 25 seconds per iteration. Today's computers would require about one second per iteration with this algorithm. Using the 48 contiguous U. S. states as data-containing polygons takes considerably less time since there are fewer areas (only 48 cells) to be evaluated.

**The Computer Programs**

The exact date of the Harvard presentation is not available, but Howard Fisher retired from his position as lab direction in 1967 so it must have been before that. Three computer versions then existed; one program treated data given by latitude/longitude quadrangles, another was for irregular polygons, and the third assumed that the geographical data are represented by a mathematical equation. The lattice version worked on the latitude/longitude grid (or any orthogonal lattice) and produced a table of x, y coordinates for each unit of latitude and longitude. In other words the program produced tables of $x = f(\varphi, \lambda)$, $y = g(\varphi, \lambda)$. These are the two equations required to generate and define a map projection. An example is given in the table (Table 1). Using a standard map projection program it is possible to plot coastlines, state boundaries,

Table 1

Map projection coordinates for a cartogram of the United States

| LONGITUDE | LATITUDE | 24 N | 29 N | 34 N | 39 N | 44 N | 49 N |
|---|---|---|---|---|---|---|---|
| 125 W | X | 0.901 | 0.512 | 0.258 | 0.916 | 1.469 | 0.705 |
| | Y | 0.520 | 1.295 | 6.529 | 17.856 | 23.559 | 28.786 |
| 120 W | X | 2.572 | 1.744 | 1.428 | 3.050 | 3.350 | 3.772 |
| | Y | 0.272 | 1.151 | 7.256 | 16.240 | 22.415 | 29.166 |
| 115 W | X | 5.364 | 5.634 | 5.422 | 4.351 | 4.779 | 5.505 |
| | Y | 0.384 | 1.203 | 7.656 | 14.756 | 22.841 | 27.355 |
| 110 W | X | 6.457 | 6.526 | 6.196 | 5.171 | 6.072 | 6.231 |
| | Y | 0.519 | 1.643 | 9.624 | 15.592 | 23.570 | 27.464 |
| 105 W | X | 7.124 | 7.118 | 6.897 | 6.082 | 6.783 | 7.151 |
| | Y | 0.526 | 1.908 | 10.340 | 16.850 | 23.989 | 27.529 |
| 100 W | X | 8.375 | 8.474 | 8.035 | 7.875 | 8.102 | 7.985 |
| | Y | 0.177 | 1.526 | 9.030 | 17.452 | 23.991 | 27.597 |
| 95 W | X | 12.221 | 13.372 | 10.826 | 10.653 | 9.944 | 10.867 |
| | Y | 0.205 | 1.873 | 9.683 | 17.012 | 24.102 | 28.341 |
| 90 W | X | 16.575 | 16.741 | 17.082 | 14.738 | 15.994 | 17.012 |
| | Y | 0.481 | 1.492 | 8.323 | 16.218 | 25.606 | 28.773 |
| 85 W | X | 23.867 | 24.124 | 23.866 | 24.013 | 24.001 | 24.279 |
| | Y | 0.325 | 1.401 | 6.476 | 14.864 | 26.785 | 28.610 |
| 80 W | X | 33.333 | 33.993 | 32.520 | 32.508 | 33.131 | 32.813 |
| | Y | 0.618 | 2.031 | 4.538 | 13.440 | 26.556 | 28.511 |
| 75 W | X | 41.886 | 42.043 | 43.210 | 40.877 | 39.962 | 39.961 |
| | Y | 2.195 | 2.971 | 4.184 | 8.479 | 26.184 | 28.638 |
| 70 W | X | 46.932 | 46.854 | 47.618 | 48.131 | 48.249 | 47.535 |
| | Y | 2.085 | 2.958 | 5.590 | 10.501 | 26.452 | 29.172 |
| 65 W | X | 48.652 | 48.117 | 48.811 | 49.469 | 49.630 | 49.291 |
| | Y | 1.311 | 2.718 | 6.350 | 13.588 | 25.608 | 28.695 |

Values in degrees

rivers, etc., by interpolation using these coordinates. Today most map projections are obtained directly from known equations, Robinson's projection being a notable exception requiring both a table lookup and an interpolation. And most map projections are not obtained by iteration; but Mollweide's projection is done in this manner to produce the table from which the projection is then calculated and interpolated. An advantage of presenting the cartogram as a table of geographic to rectangular coordinates is that this gives the final results of the iteration in a tabular form that can be passed on for the use of others, or to plot additional detail. All that is required for others to reproduce the cartogram is a simple interpolation routine. The figures (14 and 15) show an example produced from the foregoing table at the University of Michigan using a plotter .

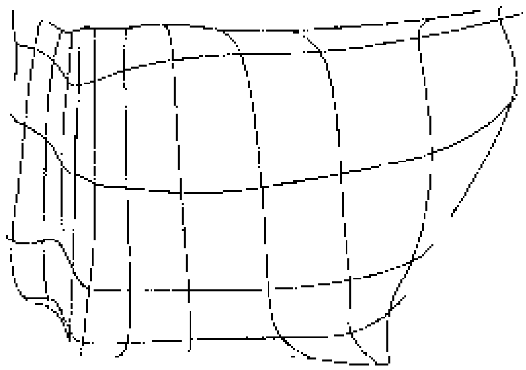


Figure 14:

The latitude-longitude graticule for a cartogram of the United States. Computed by the author.

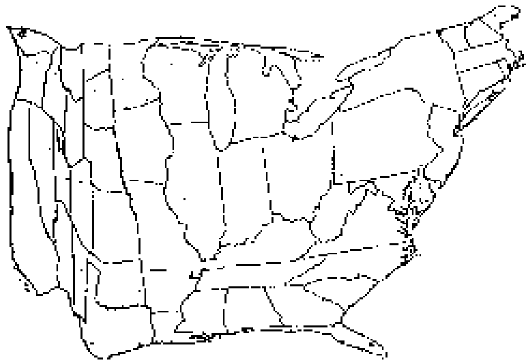


Figure 15:

The United States drawn according to the graticule of figure 14. Computed by the author

The cartogram program was also tested by entering the spherical surface area for latitude/longitude quadrangles and, as expected, yielded an equal-area map projection for the United States similar to that by Albers. As far as I am aware no one else has tested their algorithm in this manner. All equal area maps are a special case of cartograms in which the spherical surface area is the property to be preserved, as is easily seen from the equations.

A separate program, again independent of any particular map projection, computed the linear and angular distortion of Tissot's (1881) indicatrix by finite differences from this same coordinate file (Table 2). Tissot's measure of area distortion (his S) on map projections had been shown (Tobler 1961) to be equal to the desired area on a cartogram. Just as an infinite number of equal area map projections are possible there are many possible value-by-area cartograms. The choice depends on the additional conditions invoked. My choice was to come as close as possible to the conventional map by minimizing angular distortion. The Tissot evaluation program provides an objective measure as to how well this target is achieved.

The second cartogram program used state outlines, or general polygons instead of a grid, directly. Both programs allowed one to specify particular points to be fixed – not to be moved – so that, for example, some exterior boundaries could be fixed. Or points critical for recognition of landmarks could be retained. Both programs also had an option to produce an initial pseudo-cartogram (Tobler 1986a), treating the quantity to be preserved, h (u, v), as if it could be approximated by a

Table 2
Tissot's measures of map projection distortion
For a Cartogram of the USA

| Lat | Lon | a | b | k | h | 2ω |
|---|---|---|---|---|---|---|
| 29 | -120 | 0.99697 | 0.38274 | 0.38725 | 0.99522 | 52.8709 |
| 29 | -115 | 1.27582 | 0.29539 | 0.33808 | 1.26517 | 77.2165 |
| 29 | -110 | 1.20178 | 0.21477 | 0.26406 | 1.19192 | 88.3369 |
| 29 | -105 | 1.06845 | 0.23102 | 0.23704 | 1.06713 | 80.2467 |
| 29 | -100 | 1.01472 | 0.36120 | 0.36165 | 1.01456 | 56.7153 |
| 29 | -95 | 1.17550 | 0.86628 | 0.92103 | 1.13310 | 17.4216 |
| 29 | -90 | 1.75168 | 1.11213 | 1.63623 | 1.27593 | 25.8085 |
| 29 | -85 | 1.96759 | 1.36739 | 1.96237 | 1.37486 | 20.7360 |
| 29 | -80 | 1.82679 | 1.50669 | 1.82620 | 1.50741 | 11.0206 |
| 29 | -75 | 2.02536 | 1.71991 | 1.72856 | 2.01798 | 9.3559 |
| 29 | -70 | 1.87223 | 1.10027 | 1.10735 | 1.86805 | 30.1049 |
| 34 | -120 | 1.56400 | 0.44755 | 0.55812 | 1.52804 | 67.4246 |
| 34 | -115 | 1.52253 | 0.25186 | 0.26751 | 1.51986 | 91.4696 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 34 | -110 | 1.41802 | 0.31432 | 0.40560 | 1.39466 | 79.1535 |
| 34 | -105 | 1.38388 | 0.32354 | 0.39587 | 1.36495 | 76.7817 |
| 34 | -100 | 1.49628 | 0.44266 | 0.44323 | 1.49611 | 65.8309 |
| 34 | -95 | 1.46455 | 0.80596 | 0.84110 | 1.44466 | 33.7238 |
| 34 | -90 | 1.86459 | 1.47860 | 1.63220 | 1.73172 | 13.2598 |
| 34 | -85 | 2.26480 | 1.92408 | 2.16948 | 2.03094 | 9.3309 |
| 34 | -80 | 2.52269 | 1.79407 | 2.17508 | 2.20265 | 19.4347 |
| 34 | -75 | 2.41680 | 1.66779 | 1.91753 | 2.22385 | 21.1328 |
| 34 | -70 | 2.21083 | 0.96037 | 1.20576 | 2.08715 | 46.4467 |
| 39 | -120 | 1.53085 | 0.64259 | 0.68012 | 1.51454 | 48.2452 |
| 39 | -115 | 0.98584 | 0.62055 | 0.68503 | 0.94218 | 26.2880 |
| 39 | -110 | 1.43964 | 0.21640 | 0.39408 | 1.40147 | 95.2331 |
| 39 | -105 | 1.49638 | 0.23560 | 0.24867 | 1.49426 | 93.4289 |
| 39 | -100 | 0.95349 | 0.49528 | 0.51386 | 0.94360 | 36.8755 |
| 39 | -95 | 1.59407 | 1.09007 | 1.16768 | 1.53812 | 21.6453 |
| 39 | -90 | 1.90122 | 1.25518 | 1.72668 | 1.48616 | 23.6209 |
| 39 | -85 | 2.08889 | 1.27772 | 2.04533 | 1.34635 | 27.8847 |
| 39 | -80 | 2.52611 | 1.10685 | 2.50652 | 1.15052 | 45.9913 |
| 39 | -75 | 1.94916 | 0.55709 | 1.94746 | 0.56301 | 67.4824 |
| 39 | -70 | 0.94526 | 0.53746 | 0.77273 | 0.76503 | 31.9279 |
| 44 | -120 | 1.01444 | 0.55491 | 0.91442 | 0.70771 | 34.0527 |
| 44 | -115 | 0.74517 | 0.64821 | 0.66829 | 0.72722 | 7.9806 |
| 44 | -110 | 1.04047 | 0.16828 | 0.53028 | 0.91087 | 92.3680 |
| 44 | -105 | 1.33361 | 0.22041 | 0.92894 | 0.98191 | 91.5060 |
| 44 | -100 | 0.89847 | 0.85648 | 0.86942 | 0.88595 | 2.7421 |
| 44 | -95 | 1.65079 | 0.58353 | 1.46564 | 0.95787 | 57.0667 |
| 44 | -90 | 1.49627 | 0.78560 | 1.49614 | 0.78584 | 36.2918 |
| 44 | -85 | 2.39956 | 0.61478 | 2.39948 | 0.61510 | 72.6117 |
| 44 | -80 | 2.50103 | 0.39753 | 2.50058 | 0.40034 | 93.0551 |
| 44 | -75 | 1.79856 | 0.18823 | 1.79253 | 0.23894 | 108.2926 |

Tissot's 'a' measures the maximum linear stretch, 'b' is the minimum linear stretch, 'h' and 'k' are the linear stretch along the meridians and parallels, '2ω' is the maximum angular distortion. The areal distortion (Tissot's S) is the product of 'a' times 'b' and is, by design, proportional to the population. Other properties can be computed from these basic entities.

separable function of the form $h_1(u)\ h_2(v)$. To do this the program "integrated" the data in the u direction and then separately in the v direction. The effect is somewhat like using a rolling pin in two orthogonal directions to flatten a batch of bread dough. This could be used to compute a beginning configuration from which to initiate the iterations, saving computer time, but it also would affect the resulting appearance of the cartogram. A similar residual effect can be observed if a cartogram is begun with information given on any map projection of the area of interest. For example, the results are affected if plane coordinates from the Plate Carée (rectangular) projection, the sinusoidal projection, or that of Carl Mollweide, is used as the initial configuration from which to compute a cartogram of the world, or any part thereof. Using Tissot's results, integrated over the entire map, one can distinguish between, and rank, these alternative cartogram configurations.

Another trick is to begin with a simplified set of polygon outlines, iterate to convergence, and then restart with more detailed polygons. Furthermore, since data given by political units (polygons) often vary dramatically from one polygon to the next it also makes sense to use pycnophylactic reallocation of the data, which does not change the total values within any polygon but redistributes it to obtain a smoother arrangement. Thus there is less drastic fluctuation from polygon to polygon. If using finite differences (Tobler 1979) for this smooth reallocation, the individual polygons are partitioned into small quadrangular cells and the computer version for a regular lattice is used. The plane coordinates of the cell vertices are then retained for subsequent plotting. If finite elements with triangles (Rase 2001b) are used to implement the smoothing reallocation within the polygons then a computer program version for irregular areas, perhaps specialized for triangles, can be used.

In addition to the Harvard lecture the same algorithm was also presented to a conference on political districting (Tobler 1972) and at a conference on computer cartography applied to medical problems (Tobler 1979). A description of how to proceed when the geographical arrangement of phenomena is described by an approximating mathematical equation was also given in the 1961 thesis, in the 1963 paper, in the 1974 program documentation, and in the 1979 paper. An equation in two variables can depict a geographical distribution with a high degree of accuracy, or, in a simpler version, can be used to just describe an overall trend; either representation is appropriate for the construction of a cartogram. In the case of an approximating trend a the exact value-by-area property may not be obtained, since only the trend is represented by the map, and the cartogram does not retain correct area values as desired; it only approximates these. The approximation depends on the fit of the trend to the data. When such a descriptive equation is available it is no longer necessary to use an iterative procedure; the cartogram can be computed directly, depending on the complexity of the equation. For simple examples and a related idea see Monmonier (1977).

Additional programs at that time could produce a hexagonal grid to cover a region and to produce an inverse transformation from the latitude/longitude grid and the tables of x, y coordinates. The resulting tables are then for $\varphi = f^{-1}(x, y)$ and $\lambda = g^{-1}(x, y)$. This allows the inverse transformation of the hexagonal grid to lie, in warped form, over the original image, in an attempt to replicate Christaller's central place theory in a domain of variable population. This is illustrated in Figure 16. Only one of the competing computer algorithms (see below) are known to allow complete inversion of a cartogram.

The several FORTRAN programs for cartograms mentioned above were distributed by me in a 110 page Cartographic Laboratory Report (No. 3, 1974) from the Geography Department at the University of Michigan. Several of these programs were later (circa 1978) implemented on a Tektronix 4054 in the Geography Department at the Santa Barbara campus of the University of California.

An interactive version of the polygon program version, using a Tektronix display terminal, was prepared in 1970 by Stephen Guptill and myself, and presented at a computer conference (Tobler 1984). This program allowed the topological checking to be performed visually, thus avoiding the need for the tedious triangle inside-outside computation by computer. The discrepancy in each polygon was displayed on an interactive computer screen and was indicated by scaled plus or minus symbols at the polygon centroids. The proposed change at each iteration could also be superimposed on the previous result, or on the original configuration, shown as a set of dashed lines. Using an interactive cursor, the cartographer could zoom in on a location on the map to move and improve offending, or inelegant, node displacements. No numerical calculations were required by the cartographer. The computer costs were then reduced 100-fold since the topological checking could be disengaged. This was followed by further iterations, and interaction, to improve the fit to the desired areal

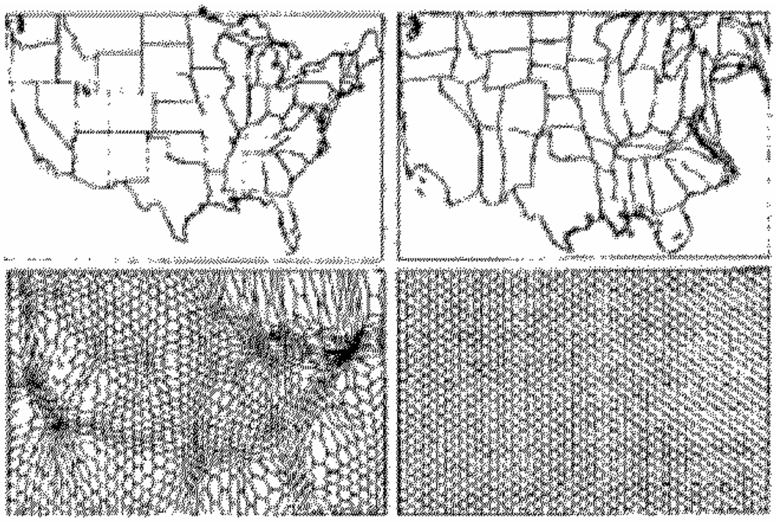


Figure 16:
Illustrating the transform-solve-invert paradigm. The bottom two diagrams are intended to be superimposed on the top two. At the top right is a population cartogram of the U.S.

Below this is a uniform set of hexagons. Using the inverse transformation the hexagons are converted to cover the United States, approximately partitioning this area into zones of equal population. Tissot's results are valid in both directions, consequently the inverse hexagons approximate his indicatrix. After Tobler (1973, “A Continuous Transformation Useful for Districting”, *Annals*, New York Academy of Science, 219 (1973), 215-220. Used by permission of the New York Academy of Sciences

distribution. An example is shown in the figure (Figure 17) for a portion of a map of South America. Williams (1978) and Torguson (1990) also developed interactive cartogram programs.

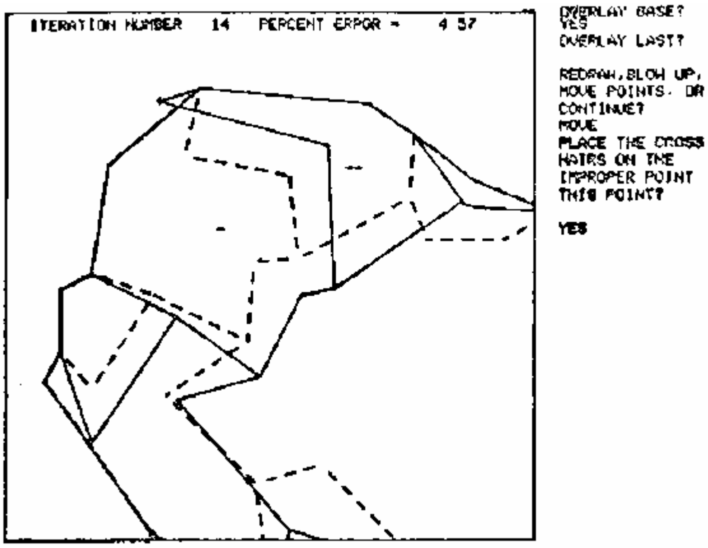



Figure 17:

One frame from an interactive cartogram program. The different lines represent stages of the iteration. The cursor placement (plus sign) at a node indicates a location for potential change by the cartographer. Positive and negative symbols indicate areas that are too large or too small relative to the desired objective. Computed by the author.

**From Rushton to the Internet**

In the following paragraphs several additional computer cartogram programs are briefly described or mentioned. The details for each program can be found in the citations. The intent here is only to give a short review of this literature.

In 1971 G. Rushton published a computer program that was based on a physical analogy. One may imagine that a thin sheet of rubber is covered with an uneven distribution of inked dots representing a distribution of interest. The objective is to stretch the rubber as much as necessary until the dots are evenly distributed on the sheet. This simple description is an approximate representation of the mathematical statement used for his computer program. If the dots represent the distribution of, say, population, the resulting cartogram is such that map areas are proportional to the population. When the rubber sheet is relaxed to its original pre-stretched form, hexagons previously drawn on the stretched surface can represent market areas. The implication is again to use the cartogram as a test of the Christaller theory of the distribution of cities in a landscape. Uniqueness would seem to me to depend on boundary conditions, since the cartogram process involves solving a partial differential equation.

In 1975 A. Sen published a theorem about cartograms. In effect he asserted that the least distorted cartogram has the minimal total external boundary length of all possible areal cartograms. Interestingly, this had been empirically discovered by Skoda and Robertson in 1972 while constructing a physical cartogram of Canada using small ball bearings to represent the unit quantity. Similar manual methods were reported by Hunter & Young (1968) and by Eastman, Nelson & Shields (1981).

Kadman and Shlomi, in 1978, introduced the idea that a map could be expanded locally to give emphasis to a particular area. This was based on rather *ad hoc* notions of the importance of an area and not on the matching to a particular distribution. Lichtner (1983) introduced similar concepts as did Monmonier (1977). Snyder's (1987) "Magnifying Glass" map projection uses a similar idea. Hägerstrand (1957) already did this of course, but he based his map distortion on an anticipated distribution, and distance decay, of the location of migrants from a city in Sweden. This resulted in an azimuthal projection with enlargement at the center of interest. More recent versions of the concept appear in Rase (1997, 2001a), Sarkar and Brown (1994), and in Yang, Snyder, and Tobler (2000). An internet firm (Idelix.com) now supports an interactive version of this technique with variable moving morphing windows.

In 1983 Appel, et al, of IBM patented a cartogram program that worked somewhat like a cellular automaton. Areas were represented as cells of a lattice and "grew" by changing state (color) depending on the need for enlargement or contraction.

A decade after my several computer programs were distributed Dougenik, Chrisman, and Niemeyer of Howard Fisher's Harvard Computer Graphics Laboratory published (1985) an algorithm that differed from the one that I had developed in only a small, but important, respect. In my algorithm the displacements to all nodes were applied only after

they had all been computed, and adjusted, for all polygons. That was at the end of one complete pass through the program. Another iteration of the same procedure then followed until the stopping criterion was satisfied. Instead of this the Harvard group, after computing 'forces' for only one polygon applied them immediately to all nodes of all of the polygons. But these displacements, discounted by a spatially decreasing function away from the centroid of the polygon in question, were applied to all of the nodes of all polygons simultaneously. Thus the objective was approximately satisfied for only one polygon, but all polygons were affected by the 'forces' and modified. They then moved on to the next polygon and repeated the procedure. Iterations were still required, with a stopping rule. The result is a continuous transformation of a continuous transformation, and that of course is continuous. As a consequence they avoided most of the necessity for a tedious topological constraint and virtually all topological problems were avoided. Depending on the complexity of the polygon shapes occasional overlapping might still occur, but only infrequently. Thus there is an improvement in speed, but not necessarily in accuracy. Almost all subsequently developed computer programs stem from this 1985 publication, which included pseudo-code.

In 1988 Tikunov of the Soviet Union presented a brief review of the history of cartograms and described several manual, mechanical, and electrical methods of production for these types of maps, along with a sketch of a computer method. This paper also included many references to the considerable Russian literature. A new algorithm was also presented in 1993 by the Soviet authors Gusein and Tikunov. Another version, with emphasis on medical statistics, formed the basis for a Ph.D. dissertation at the University of California at Berkeley by Merrill (described in Merrill et al. 1991, also Selvin et al. 1988). In these medically oriented papers the important emphasis is on the analytical use of a cartogram as a new geometric space in which to do statistical testing. The objective hence is not visual display but analysis. An earlier health related paper, using a manual procedure and with a similar objective is by Levinson & Haddon (1965).

D. Dorling, in 1991, developed a novel approach. Using only centroids of areas he converted each polygonal area into a small bubble - a two dimensional circle. These bubbles are then allowed to expand, or contract, to attain the appropriate areal extent. At the same time they attempt to remain in contact with their actual neighbors. Dorling also colored the resulting circles depending on some additional attribute. His later (Dorling 1996) work contains a rather detailed history of the subject and gives numerous examples of alternative types and, as of the 1991 date, includes the most comprehensive bibliography on the subject of area cartograms, along with two programs to compute cartograms. One of these programs was for polygons converted to a raster, and included an inverse procedure. The other program was for his bubble algorithm. This is most popular in the United Kingdom, perhaps because the complete computer program was published there, and now also seems to be available on the Internet. He later (1995) published an atlas of social conditions making extensive and effective use of the display capabilities of his procedure. Dorling did not present any equal area maps using this algorithm, with spherical land area ($km^2$) as the property to be preserved, except as initial configurations.

Adrian Herzog of the Geography Department of the University of Zürich has also prepared a program for interactive use on the Internet (http://www.statistik.zh.ch/map/mapresso.htm). Two further implementations for use in conjunction with a GIS from ESRI have also been reported (Jackel 1997, Du 1999). An undergraduate thesis has also been presented by Inglis (2001), and there is a master's thesis by Torguson (1990). Some literature exists in German (Elsasser 1970, Kretschmar 2000, Rase 2001a) and from France too (Cauvin and Schneider 1989).

A recent search on Google located 2,170 entries under ‘cartogram’ on the internet, including analytical applications in medicine (mostly epidemiology), uses for display in geography, and, interestingly, a large number of exercises for grade school children. Also included on the internet are animations, made possible by the iterative nature of the algorithms An example is at http://www.bbr.uni.edu/cartograms.

**Recognition Difficulties**

It has been suggested that cartograms are difficult to use, although Griffin (1980, 1983) does not find this to be the case. Nevertheless Fotheringham, et al (2000, p. 26), in considering Dorling’s maps, state that cartograms

“*…can be hard to interpret without additional information to help the user locate towns and cities”.*

The difficulty here is that many people approach cartograms as just a clever, unusual, display graphic rather than as a map projection to be used as an analogue method of solving a problem, similar in purpose to Mercator’s projection. Mercator’s map is not designed for visualization, and should not be used as such.

If the anamorphic cartograms are approached as map projections then it is easy to insert additional map detail. In the case of Dougenik et al’s, or Dorling’s, or other, versions simply knowing the latitudes and longitudes of the nodes or centroids allows one to draw in the geographic graticule, or to display any additional data given by geographic coordinates. And this can be done using a standard map projection program augmented by a subroutine to calculate from a map projection given as a table of coordinates. Included here, in addition to the geographic graticule, could be roads, rivers, lakes, etc., which would enhance recognition. Dorling’s bubbles could be replaced by boundaries of larger administrative units, or other features. Thus the 'bubbly' effect need not be retained. One could then even leave off of the map the administrative or political units that were used in the construction of the cartogram. And one could replace these by superimposing an alternate set of boundaries or other information (for example, disease incidence, poverty rates, roads, or shaded topography). This requires a bit of simple interpolation from the known point locations. It is also possible to lightly smooth the latitude and longitude graticule obtained to avoid inopportune kinks introduced by some algorithms.

It has recently been proposed that “brushing” techniques, borrowed from statistical graphics, can be used to overcome the problem of difficult geographical recognition on areal cartograms. In this procedure a normal map is presented alongside of the cartogram on a computer screen and by pointing at a location on one of the two maps the comparable position of the other map is highlighted. Several implementation of this procedure can be found on the internet. In this context a point-wise inverse mapping is available. Of course the brushing technique only works on an interactive screen. To my knowledge the efficacy of this method has not been studied.

As observed above, it is also possible to use Tissot’s results to calculate the angular and linear distortion of the map. The areal distortion, Tissot’s S, has been shown to be equal to the distribution being presented. Tissot’s indicatrix (Robinson 1951) is useful in comparing two cartograms obtained from the same data using alternative algorithms. If they both preserve the value-by-area property then the measure of angular distortion (Tissot's $\omega$) is important. The sum, or integral, of the indicatrix properties over the entire map can be used to provide an overall, i.e. global, measure of the distortion. A similar result holds true of cartograms that stretch space using time or cost distance rather than areal exaggeration. Sen (1976) also presents some possible alternative measures of distortion.

It is now also apparent that one can calculate a cartogram to fit on a globe - a mapping of the sphere onto itself - but this has as yet not been done.. It is not necessary to invoke a plane map at all. Examples might be to represent surface temperature or annual precipitation, constructing the globe gores in the usual fashion. From such a representational globe <u>any</u> conventional equal area map projection (Albers, Mollweide, sinusoidal, Lambert, etc.) can be used to represent the information as a proper (but different, depending on the projection) areal anamorphose on a flat map. Satellite image globes have to some extent supplemented political globes and anamorphic globes might someday also be constructed.

**Extending the Concept**

Computer scientists, including Edelsbrunner & Waupotisch (1995), have also studied the problem. In general they conceive of area cartograms as having a graphical display function - yielding insight into some problem - rather than as an analytical tool, and not a graphical nomograph for the solving of a specific problem. A recent implementation is from Texas, again in a thesis (Kocmoud 1997, also Kocmoud & House 1998). This interesting algorithm attempts to maintain shape in addition to proceeding to correct areal sizes. The shape preservation alternates with area adjustment in each iteration, neither being completely satisfied. Somewhat similar, but considerably faster, programs and algorithms have recently been developed by researchers at the Martin-Luther University in Halle; (Panse 2001) and the AT&T Shannon Research

Laboratory (Keim, North, and Panse 2002; Keim, North, Panse, and Schneidewind 2002). These papers also review, though not always correctly, and display copies of existing alternative cartograms.

It is important to recognize that these computer engineers have employed a different objectivein addition to the correct preservation of the correct topology. Instead of an areal cartogram, *senu strictu*, they attempt to "balance" the value-by-area concept with shape preservation. That is, they allow departure from the objective of exactly fitting the areal distribution of concern to better conserve shapes. They introduce two finite measures of departure from the objective. One applies to area, the other to shape. Contrast this with my objective of faithfully preserving the areal distribution and doing this in a manner that then tries to best preserve shape, without giving up the precise value-by-area property so necessary for the theoretical objective. This engineering approach is reminiscent of the Astronomer Royal George Airy's (1861) projection by a "balance of errors". Airy, recognizing that a geographic map could not be equal area and conformal at the same time, chose to give up an exact fit to either of these important properties and instead used a least squares approach to find a compromise map projection. Aims comparable to those of Airy have been invoked for map projections, with slightly different mathematical objective functions, by Jordan (1875), Kavraisky (1934), and others, and are described by Frolov (1961), Biernacki (1965), Mescheryakov (1965), and Canters (2002). Tissot's areal distortion (S) on a value-by-area cartogram (also equal to his a*b, the product of the indicatrix axes) has been shown to be equal to h (φ, λ), the areal density of concern, therefore a new criterion, comparable to that of Airy, can be formulated as a balancing of the value-by-area property with the property of conformality (local shape preservation). Then in order to use the least squares criterion, as did Airy, for this new objective we need to minimize the following double integral,

$$\int\int \; [ab - h(\varphi, \lambda)]^2 + [a - b]^2 \; \cos\varphi \; \partial\varphi \; \partial\lambda$$

taken over the region of concern. Some variants of this integral are also possible - differential weighting of the criteria is an example. In the case of plane maps one substitutes u, v for the latitude and longitude and drops the cosine term. Observe that this is a global, not a local criterion which instead would be used to minimize the maximum of the proposed function at all locations. The first squared term measures the departure from the value-by-area property. The second term measures the departure from conformality. The ratio of the minimum linear stretching to the maximum stretching (b / a), or the logarithm thereof, could also be used to measure departure from conformality as is done in the closely related field of quasiconformal mapping (Teichmüller 1937, Gehring 1988). The recently invented computer algorithms use finite measures of the fit to the phenomenon of concern and to the preservation of shape. This is done by an alternating iteration until some combined criterion is satisfied. They then present speedy, and recognizable, anamorphoses on which additional information may be displayed.

It would be of interest to see whether new compromise (more conventional) map projections of the world, or parts thereof, can be created using these finite algorithms, simultaneously minimizing areal and angular distortion when given information defined by polygons. A map of the contiguous United States using county areas defined by finite polygon coordinates, for example, or spherical quadrilaterals, might approximate, or balance, both Albers' equal area conic projection and Lambert's conformal conic projection for this region. Interpolation could then produce a table of map projection coordinates for general use.

Along with Tissot's measure of areal distortion for evaluating a map projection it is now possible to add a measure of the departure from density preservation, in addition to the conventional measures of linear and angular distortion. My initial objective was to have no departure from density preservation, combined with minimal departure from conformality The newly introduced versions have departures from both criteria in a 'balanced' fashion. Now the two independent measures of departure can be shown on the anamorphoses using Tissot's indicatrix or by choropleths or isolines. As long as the cartograms are used only for visual display it is clear that some departure from density preservation and from conformality can be tolerated since MacKay (1954, 1958) has shown that visual estimates of area and conformality are not terribly accurate.

**Still Needed**

As stressed in the foregoing materials, the general factors in evaluating an algorithm must be to 1) show correct value-by-area, 2) preserve shape to the extent possible, and 3) be efficient, in that order. For the last item the computational complexity of a cartogram would appear to be the same as that of any map projection. I am not aware of any results in this direction but would expect to find this to be polynomial in nature. The enhanced possibility of producing the inverse transformation also seems useful for theoretical purposes, using the important transform-solve-invert paradigm (Eves, 1980; 215-228) This is similar to the use of a conformal Zhukovskii transformation in order to study the aerodynamics of airfoils (Ivanov and Trubetskov 1994, pages 70-73).

As a final remark, none of the several algorithms presented to date are capable of replicating Raisz's 'rectangular statistical cartogram'. This seems to be because the exact, correct, topology is difficult or impossible to maintain when forcing the areas into rectangles. The adjacency graph of the original unit outlines does not agree with that of the 'rectangular' representation. This can probably be proven to be impossible, along topological lines similar to Euler's Königsberg bridge problem. Raisz actually forces his entire map (Figure 3) to fit into a rectangle, sometimes with an appended square for New England or Florida. In the last thirty five years several atlases have featured world cartograms with all countries as squares or rectangles, in their approximately correct position and size according to some property, usually population or per capita income, etc. (See Dent, op. cit. for citations). The countries are then colored according to some additional attribute. The 'square-like' countries are not generally forced into an overall rectangle. The similarity is to Levasseur's map of Europe (Figure 2). To my knowledge all of these are still produced by hand. A useful critical measure might be based on the distance of the true adjacency matrix from that as represented on the cartogram, but I am not aware of any such proposals. Only at the intersections of political boundaries, whose latitude and longitude are known, could anything like Tissot's indicatrix be calculated or could the entire graticule be interpolated to illustrate the warping of the usual map. But perhaps other possibilities exist for alternative measures for the measuring of the similarity of two cartograms, along the lines given in Sen (1976) or Tobler (1986b).

The computer construction of cartograms has progressed rapidly in the last several years. Of course it is difficult to predict the future but I expect that with the increased speed and storage capabilities of future computers the next thirty-five will lead to further changes in this field. In one sense the study of the subject of map projections has already been rendered obsolete since calculations on the sphere or ellipsoid are now easy. But display and visualization continue to be important, and this is how most cartograms are used.

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Addendum:

Shortly after publication of this article the following papers were published:

M. van Kreveld,, Spreckmann, B., 2007, "On Rectangular Cartograms", *Computational Geometry: Theory and Applications*, 37(3):175-187 The authors describe a computer method for doing Raisz-type Value-By-Area cartograms. Related publications are at www.cs.uu.nl

M. Gastner, Newman, M, 2004, "Diffusion-Based method for producing density-equalizing maps", *Proc*. National Academy of Sciences of the United States of America, 101(20):7499-7504. Their method is mathematically similar to the analogue manual procedure used by Gilliham, 1927.