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VISUALIZING THE IMPACT OF TRANSPORTATION ON SPATIAL RELATIONS

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Abstract:
Empirical measurements of spatial separations before and after transportation innovations can be considered to be changes in geometry and, as such, evaluated using Gauss's first fundamental form. These numerical quantities should be of practical benefit in evaluating transportation systems. Graphical examples using empirical road distances are given.

It is often asserted that the world is "shrinking." Clearly the overall effect of changes in transportation technology has been to reduce separations, measured in units of time or effort, and is thus a shrinking. But these effects are very unevenly distributed. A non-uniform shrinking inevitably modifies the geometrical relations within our environment, sometimes drastically so, and this has substantial impacts on geographical interactions. It is the purpose of a current research effort to study these environmental modifications and their impacts. This report is a synopsis of a portion of this research. Additional material can be found on my web site (http://www.geog.ucsb.edu/~tobler).

Changed spatial relations can be interpreted as a change of geometry. In the present context this statement is taken literally and transportation is studied as it might be done by a differential geometer. From this point of view, a transportation system is a "mapping" which converts one geometry into another. More specifically, we can trace the consequences of a transportation induced transformation on geometrical entities such as distances, areas, and angles. There are, of course, also topological changes of equal, or greater, import; in this case the mappings are no longer one-to-one and continuous. These cases are excluded in the present discussion.

As is well known, Gauss showed, in circa 1827, that the fundamental metric of a surface can be written as $dS^2 = g_{\alpha\beta}du^{\alpha}dv^{\beta}$, where I have used Einstein's summation convention to simplify the notation. After a transformation this metric tensor takes the same form, but with altered coefficients (the g's, which depend on the partial derivatives of the transformation), and the change in the coefficients allows us to describe the modification of the geometry rather completely. One author (O'Neill, 1966) refers to the $g_{\alpha\beta}$ coefficient matrix as the shape operator. An illuminating discussion is also available in Misner, et al, 1973, pp 305-309.

There are some rather obvious points of contact between classical differential geometry, and classical transportation geography. One of these is the identity between the geographical minimum time paths introduced by Francis Galton in the 1880's and their orthogonal trajectories (as used on isochronic maps), and the polar geodesic coordinates of Gauss (for which the first fundamental form is particularly simple). Apparently no one has exploited this identity in a systematic manner. There is also an observable similarity between the problems of finding a geodesic over a surface of variable curvature and the minimal path algorithm through a network (Nicholson, 1966; Schilling, 1928). But one approach is infinitesimal, the other finite. Some practical advantages might be obtained by exploiting this theoretical similarity.

In the context of transportation, our approach must be based on observation and measurement and not, as seems to be the usual case in mathematics, on assumed simple surfaces and transformations given by a priori equations. Thus we need an empirical differential geometry. The main difficulty lies in extrapolating the finite measurements, obtained at irregularly arranged positions on the earth, to form (or at least to make a reasonable approximation to) a differentiable continuum. For this purpose a non-parametric interpolation algorithm designed for the production of smooth (of class C^2) contour maps, devised by myself, is used. This procedure resembles two dimensional least squares splining. Finite difference equations are then be used to calculate the distortion tensor. The details of this interpolation procedure have been published (Tobler, 1994) and need not be repeated.

The techniques described here are explored using real transportation data, assembled over the last two decades by myself from published sources and covering several continents. The data are, for example, road kilometers versus spherical distances, or travel times by air, rail, and ship, or comparable travel costs, taken from official guides and travel brochures. In addition, travel times measured by students in cars along street segments are available. The purpose is to have data to enable studies to be made at varying levels of spatial resolution. Not all of these examples are discussed in the present report; only one is illustrated.

The techniques should be of value in practical problems of facility location, in environmental impact studies, and in yielding substantive knowledge of the role of transportation in society. More specific objectives of the research are to attempt to calculate the curvature of the geometry induced by transportation systems in a large region. I also wish to be able to compute isochrone maps centered at any point, without having to collect data from every point. Another way of stating this is to point out that the formula given earlier allows us to compute the distance in kilometers between two locations on the surface of the earth if we know their latitudes and longitudes, and the particular set of coefficients which apply to the earth. Why should one not be able to do the same for travel times, or travel costs? Such a computation could easily be done on a scientific pocket calculator, vitiating the need for rate books. The question is not whether this can be done, but how well it can be done with a simple model and a modest number of parameters. For example, there is evidence that automobile travel speeds are reduced in central cities, roughly in inverse proportion to the population density (c.f. Angell & Hyman, 1976). There are some models of the population density distribution within cities that, certainly at a national scale, would allow general statements in this regard, using only population data and without additional specific transportation measurements.

Scattered attempts have appeared in the transportation literature in which partial models along the lines suggested here. Some of these are listed in the bibliography. (Love & Morris, 1972, 1979; Trunin & Serbenyuk, 1968; Vaughn, 1987). The articles, while individually competent, interesting and useful, are very diverse and scattered, in part because transportation studies tend to be media specific, local in nature, address immediate problems, and usually consider networks rather than continua; exceptions include Angel & Hyman, 1976; Beckmann, 1952; Mayhew, 1986; Puu, 1979; Iri, 1980, pp; 263-278; Taguchi & Iri, 1982, and a few others. The present point of view is also a continuum one and is seen to have considerable merit.

Given an N by N table of travel disutilities (time, cost, road distance, etc.) between N locations one can construct several graphics to show the departures from a planar (Euclidean) geometry[1]. A common technique is to use a trilateration or a multidimensional scaling to estimate best fitting plane coordinates for the places (e.g., Mueller, 1978, 1984, Gatrell 1983). When the travel disutilities are not symmetric ($D_{ij} <> D_{ji}$), the normal empirical case, then this technique breaks down and more advanced models are needed, as outlined by Tobler, 1978 (also see Zaustinsky, 1959).

Within this symmetric framework, a simple graphic is to display the places in their correct spatial location but to then "enhance" the view of the links between them by putting in resistor-like symbols to display the travel difficulties. Another graphic device is to imagine a transportation "surface", conceptualized as a topographic-like surface over which the transportation takes place. The distance along the surface between nodes is then longer than the distance between the orthogonal projection of these nodes onto the base level. The ratio of this spatial (three dimensional) distance to the (two dimensional) base distance gives the Gaussian coefficients for this particular piece of the geometry. The entire collection of Gaussian coefficients describes the geometry of the transportation space, here assumed two dimensional. A previous report (Tobler, 1993) goes into additional detail.

These ideas are illustrated using twenty two towns in the western (west of the 105th meridian) portion of the state of Colorado, with road distances taken from the official highway map, and geographic coordinates from the Atlas of the United States. This is a rather mountainous piece of territory.

Gauss (1827) showed that the fundamental metric of a surface, specifying the geometry, can be written as

$$dS^2 = g_{\alpha\beta} du^\alpha dv^\beta = E\, du^2 + 2\, F\, du\, dv + G\, dv^2 \, .$$

The g coefficients depend on the partial derivatives of the surface. For a Monge surface (that we now have) of the form

$$X=x,\ Y=y,\ Z=z\ (x, y),$$

the Gaussian coefficients are:

$$g_{11} = \text{Gauss' } E = 1 + (\partial z/\partial x)^2$$
$$g_{12} = g_{21} = F = (\partial z/\partial x)\ (\partial z/\partial y)$$
$$g_{22} = G = 1 + (\partial z/\partial y)^2$$
$$g_{11}g_{22} - g_{12}g_{21} = EG - F^2 = 1 + (\partial z/\partial x)^2 + (\partial z/\partial y)^2$$
$$K = \quad [\ (\partial^{2z}/\partial x^2)\ (\partial^{2z}/\partial y^2)\ -\ \partial^{2z}/\partial x\, \partial y\,]\ /\ [g_{11}g_{22} - g_{12}g_{21}]$$

where K is the Gaussian curvature (Rektorys 1969:343+) These components of the Gaussian metric can be estimated from the partial derivatives of the ‘transportation surface’. These interpolated coefficients are shown in the accompanying figures.

A related consideration is to contemplate the set of all places on the surface of the earth that you could reach within one hour of where you are now. The outer edge of this set forms a geographical ‘circle’ of one hour radius. What a curious circle it is! Its circumference is hardly $2\pi r$; its area, in square hours, is not $\pi\, r^{2.}$ The circle most probably has holes in it, and probably consists of disjoint pieces when shown on an ordinary geographical map. The shape of this circle depends on the place and time of day at which you start your journey. A resident of Paris will have a geographical circle of one hour radius with a different circumference, shape, orientation, and area from that of a person in Los Angeles. This is a bit like the geometry on the surface of a cucumber; here a circle at the narrow end will differ from one near the middle. Or try drawing concentric circles on the surface of a potato from some starting point. The differential geometry introduced by Gauss in the early 1800's can handle many of these situations. Some examples are available at the URL cited earlier. These maps in effect show the system known as polar geodesic coordinates, first defined by Gauss (1827). Isochronic maps, used by Francis Galton in the 1880's, use curves depicting lines of equal time from a center. These are a special case of polar geodesic circles in which the radial lines have been omitted. Polar geodesic coordinates are a curvilinear equivalent to polar coordinates, In this coordinate system the first fundamental form

$$ds^2 = g_{11}du^2 + 2g_{12}dudv + g_{22}dv^2$$

reduces to the simpler form

$$ds^2 = dr^2 + g_{22}d\theta^2,$$

where r is the radial geodesic and θ is the circumferential coordinate. The area contained within one "circle" is now given by the extended formula

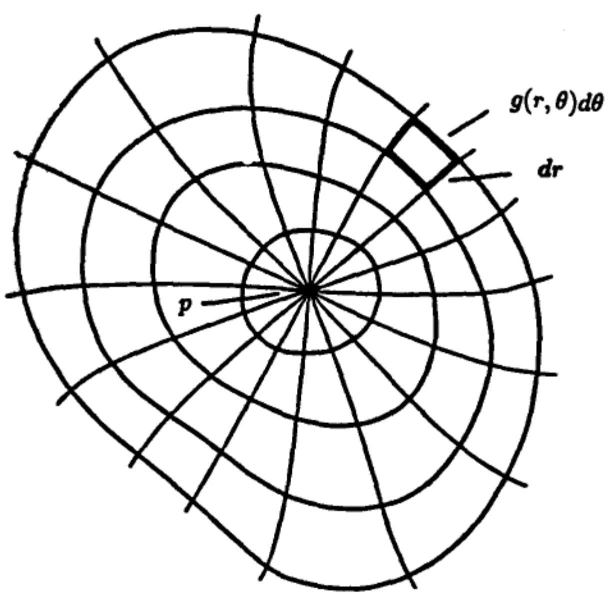
$$A = \pi r^2 (1 - \cos(r \sqrt{K})) \approx \pi r^2 - \pi K r^4/12 + ...$$

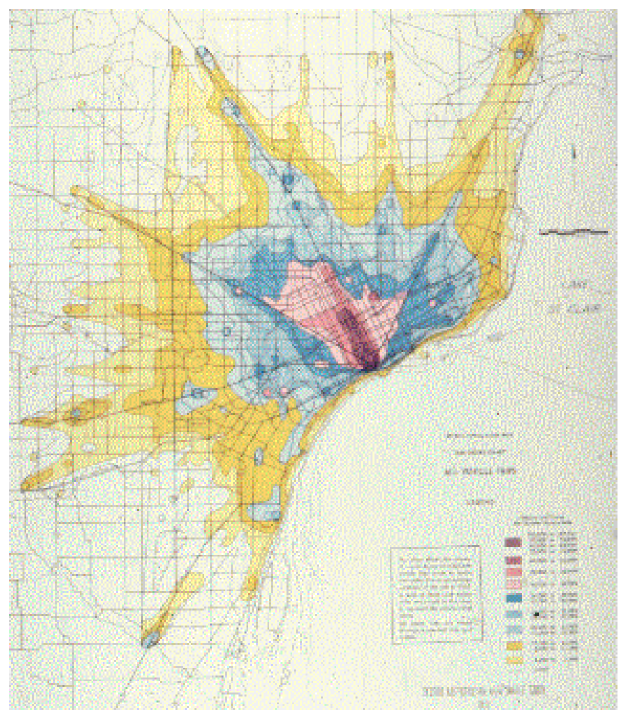
and the circumference is given by

$$C = 2 \pi r \sin(r \sqrt{K}) \approx 2 \pi r - \pi K r^3/3 + ...,$$

where K is the Gaussian curvature, $K = (-1 \partial\sqrt{g_{22}}) / (\sqrt{g_{22}} \partial r^2)$.

The relation to the better known Euclidean formulae (K=0) for area and circumference should be obvious; they constitute the special case of zero curvature. The example at the right, below, shows isochrones from the center of Detroit.





Geodesic polar coordinates.

Let us now take the formula for area and see if it is possible to compute the curvature from this. Given that $A = \pi r^2 - \pi K r^4/12$ we can solve for $K = 12(A-\pi r^2)/(\pi r^4)$, or

$$K = 12A/(\pi r^4) - 12/r^2.$$

Now we need some data and this is given in the following three tables

A) Travel times (hours) from Leipzig, 1911

B) Surface area (km^2) as measured from maps given in Riedel (1911)

C) Computed curvature

A	B	C
1	166	-12
2	571	-3
3	2064	-1.33
6	23,930	-.33
9	814306	-.15

A) Travel times from downtown Washington D.C., 5:15 p.m.

B) Surface area ($miles^2$) as measured from maps given in Wingo (1961)

C) Computed curvature

A	B	C
5	1.14	-.48
10	4.4	-.12
15	11	-.005
20	27	-.03
25	62	-.02

A) Travel times from downtown Washington D.C..,6:15 p.m.

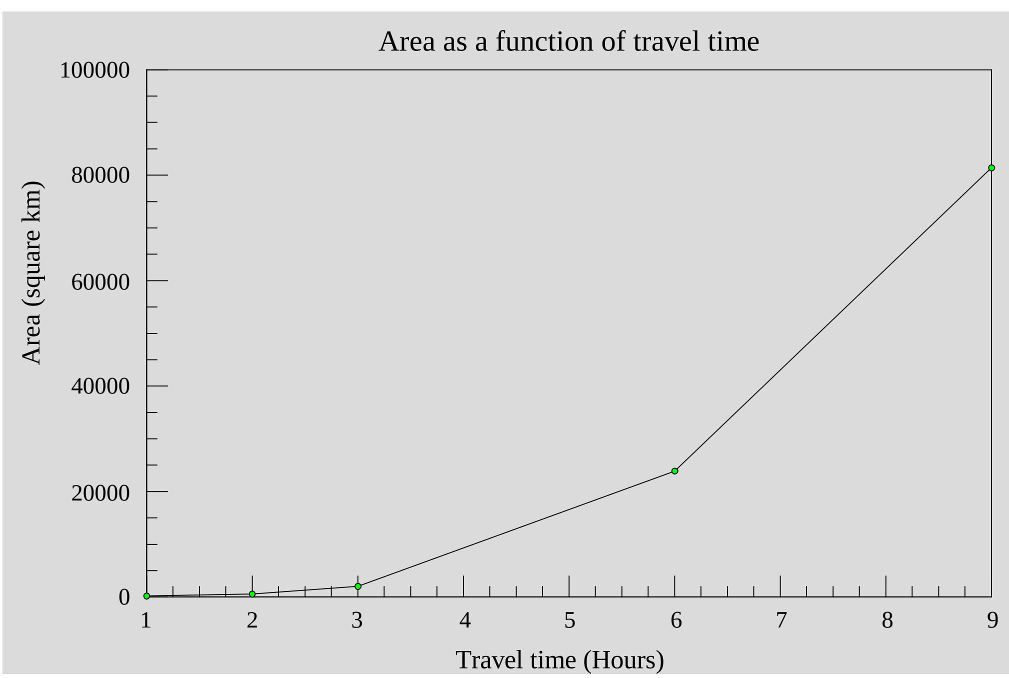
B) Surface area ($miles^2$) as measured from maps given in Wingo (1961)

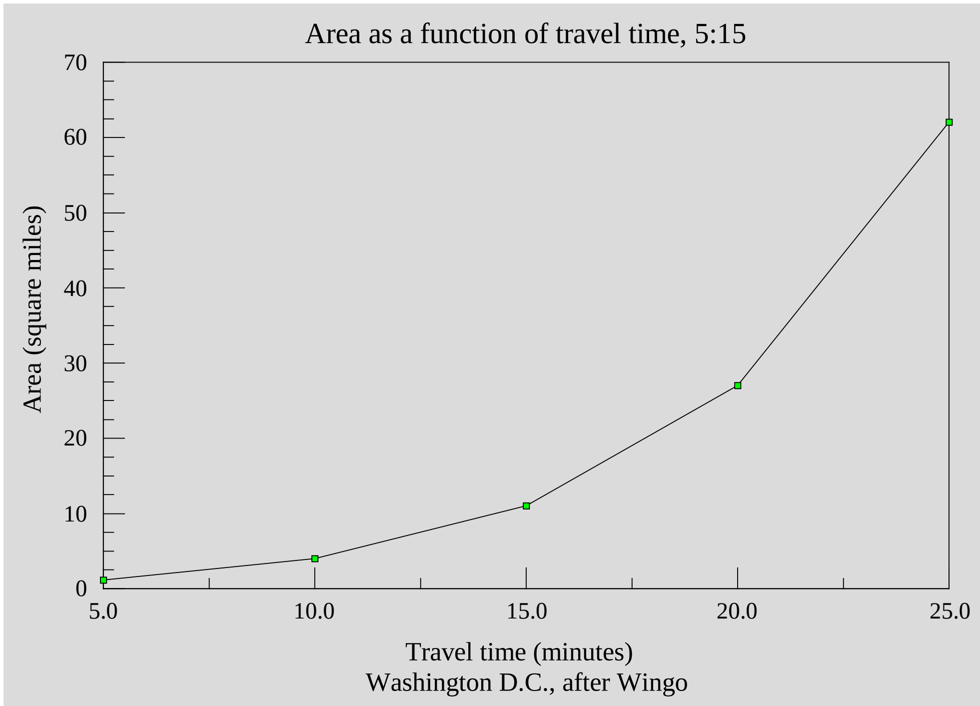
C) Computed curvature

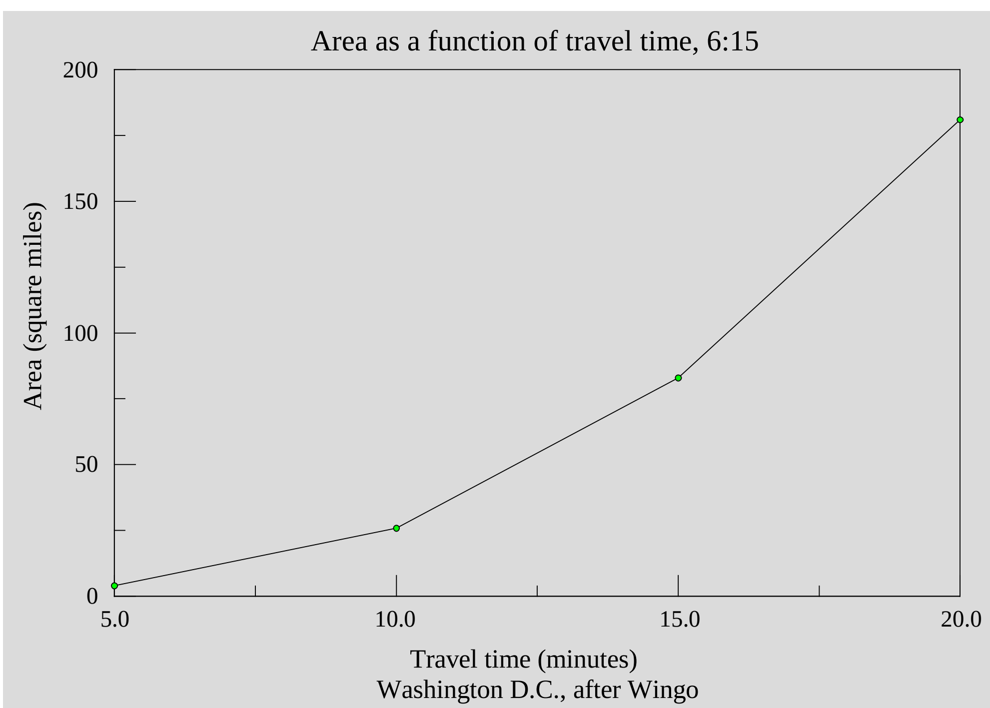
A	B	C
5	4.16	-.48
10	26	-.12
15	83	-.005
20	181	-.03

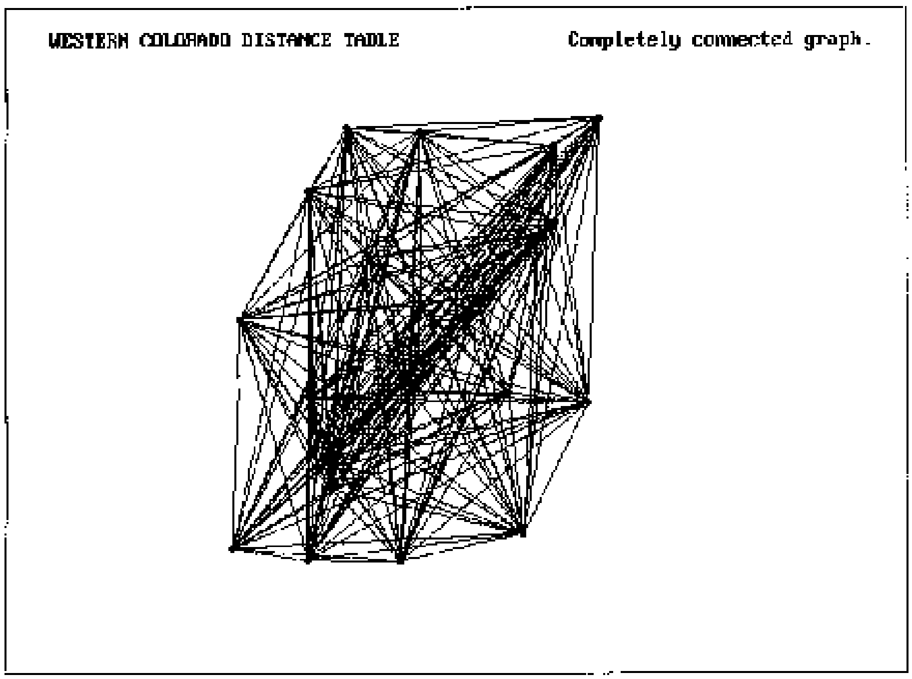
These results are astounding. The computed curvatures are all negative and, with the exception of the first two for Leipzig, very small and close to zero. And the values computed for the two different times in Washington D.C. are identical. This in spite of the fact that the covered area becomes much larger after the rush hour. If we plot graphs of the relation of the area versus the travel times, the curves clearly resemble a quadratic function: For Leipzig km^2 area = 111.8*travel $hours^{2.9}$ gives a fit of 96% (r-squared), for Washington D.C. at 5:15 p.m. the equation is square mile area = 0.19*travel minutes$^{2.43}$ (92%) and at 6:15 p.m., square mile area = 0.05*travel minutes$^{2.73}$ (99%). Of course the sample sizes are very small.

The values of curvature computed in this small experiment strongly, and quite unexpectedly, suggest a value of zero, implying a flat Euclidean geometry, Yet the irregular shape of the isochronic lines, thought of as concentric geodesic circles, but strongly direction dependant (based on the location of major highways) hardly fit this image. It seems that further examples must be assembled.




Area as a function of travel time, 5:15
70
60
50
40
30
20
10
0
Area (square miles)
5.0
10.0
15.0
20.0
25.0
Travel time (minutes)
Washington D.C., after Wingo


Area as a function of travel time, 6:15
200
150
100
50
0
Area (square miles)
5.0
10.0
15.0
20.0
Travel time (minutes)
Washington D.C., after Wingo



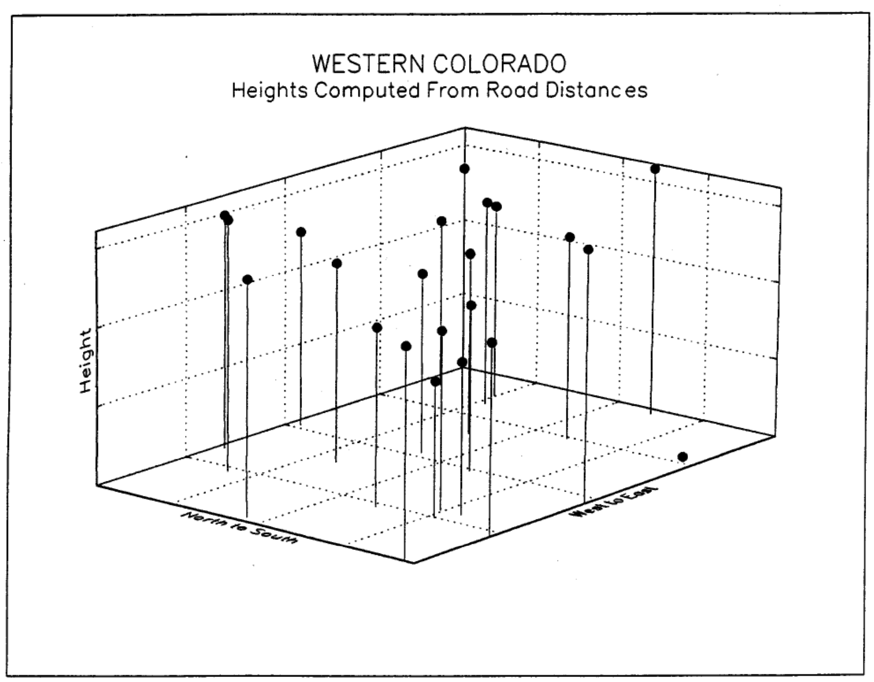
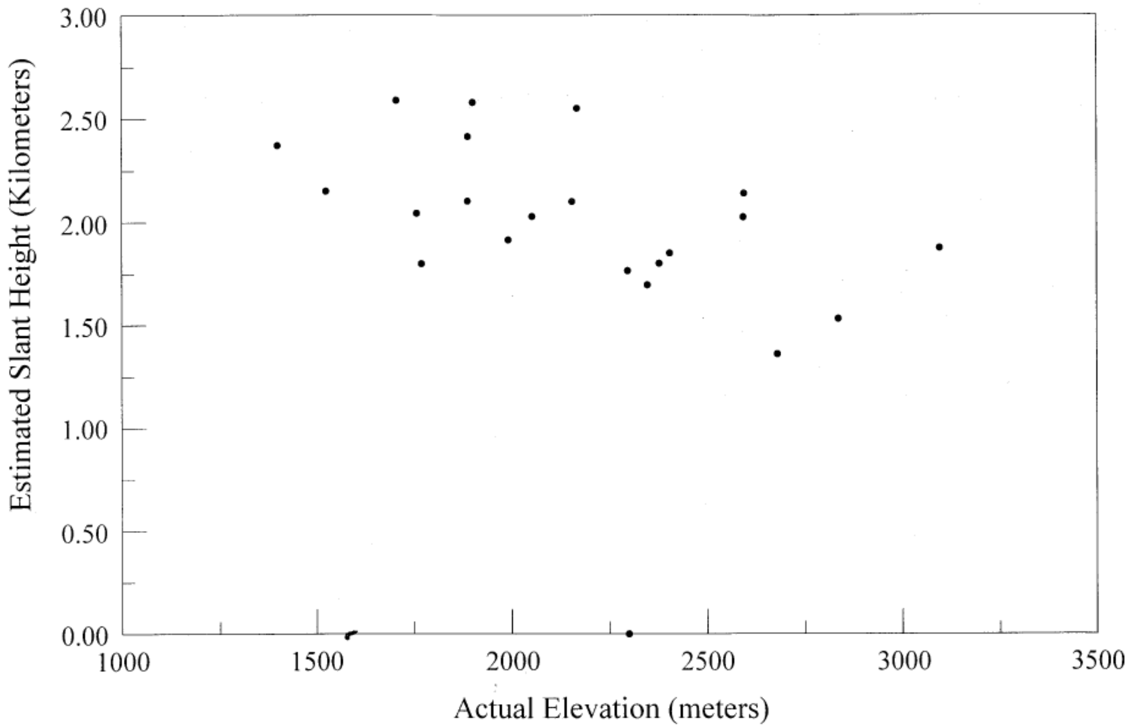
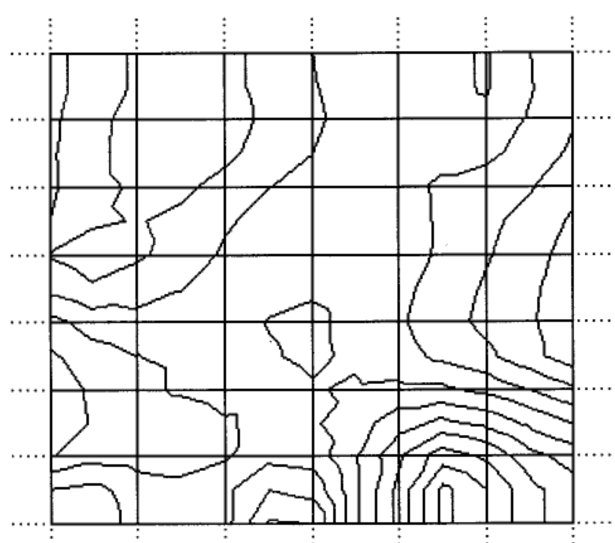


Figure 3.
Road distances approximated by raising locations to a height which yield proper, in the least squares sense, interpoint road distances when measured in three

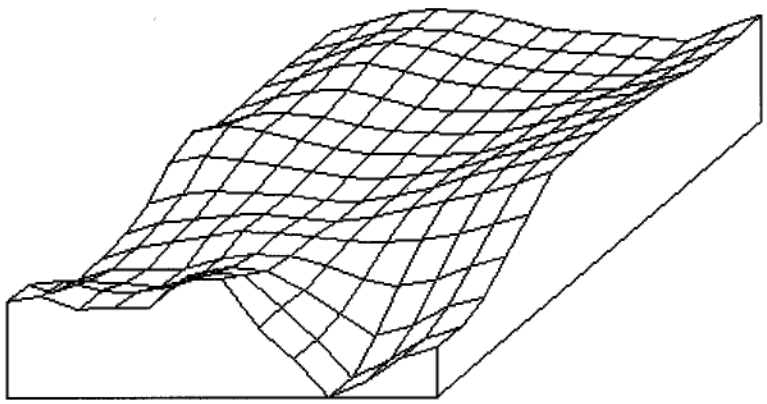
COLORADO DATA
Slant Heights Estimated from Road Distances



Colorado Heights from Slant Distances
The contour interval is .2966 The array is 15 by 13 Grid interval is 2



Colorado Heights from Slant Distances
Isometric diagram. Hidden lines not deleted.



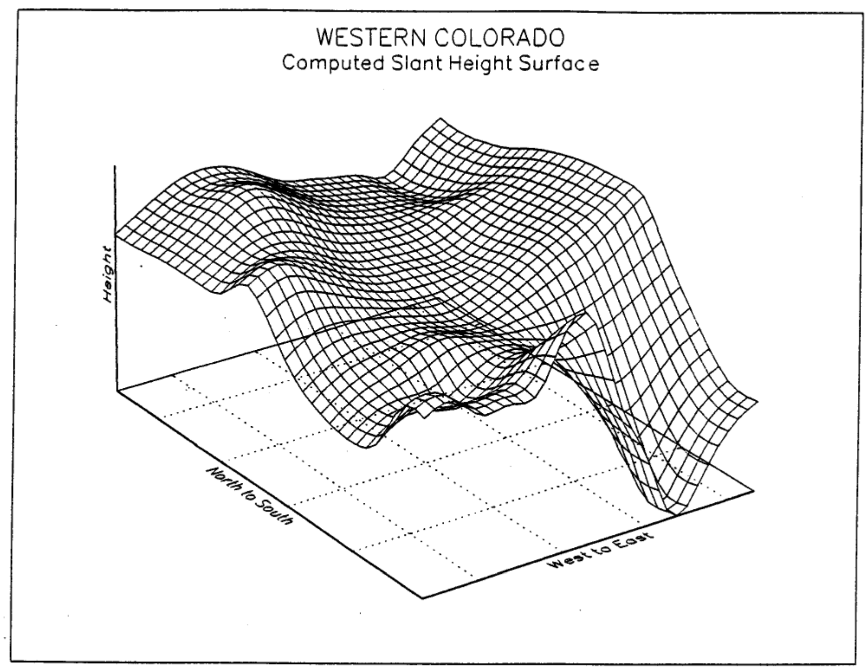
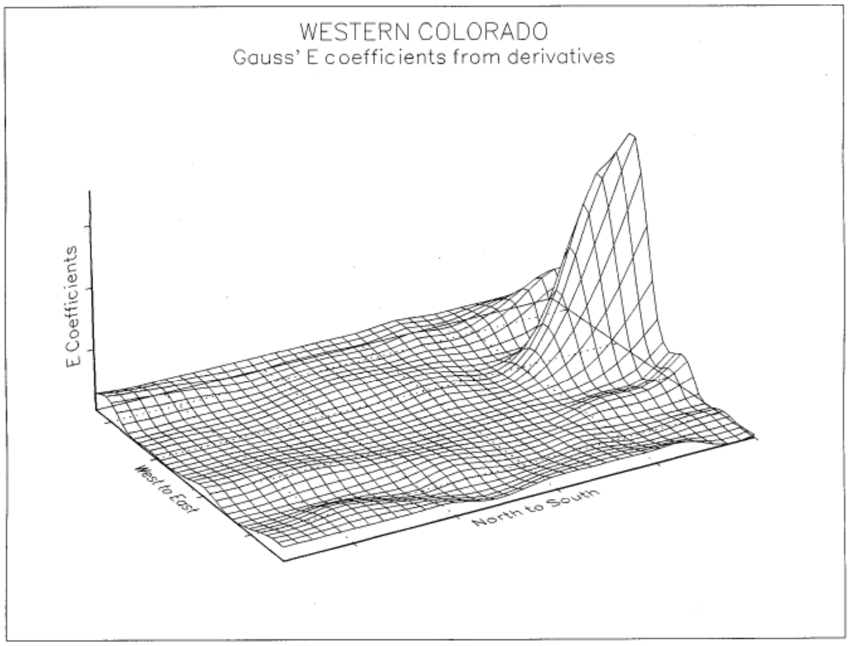
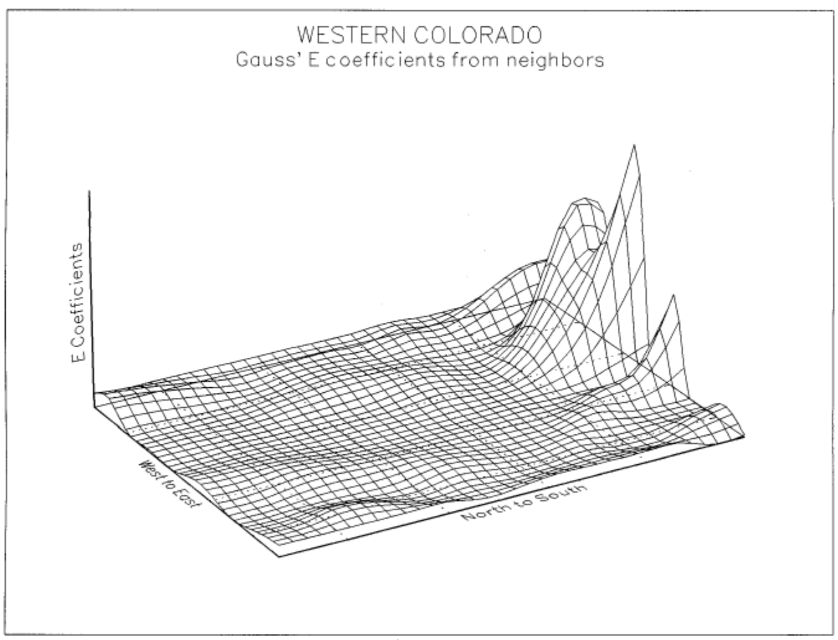
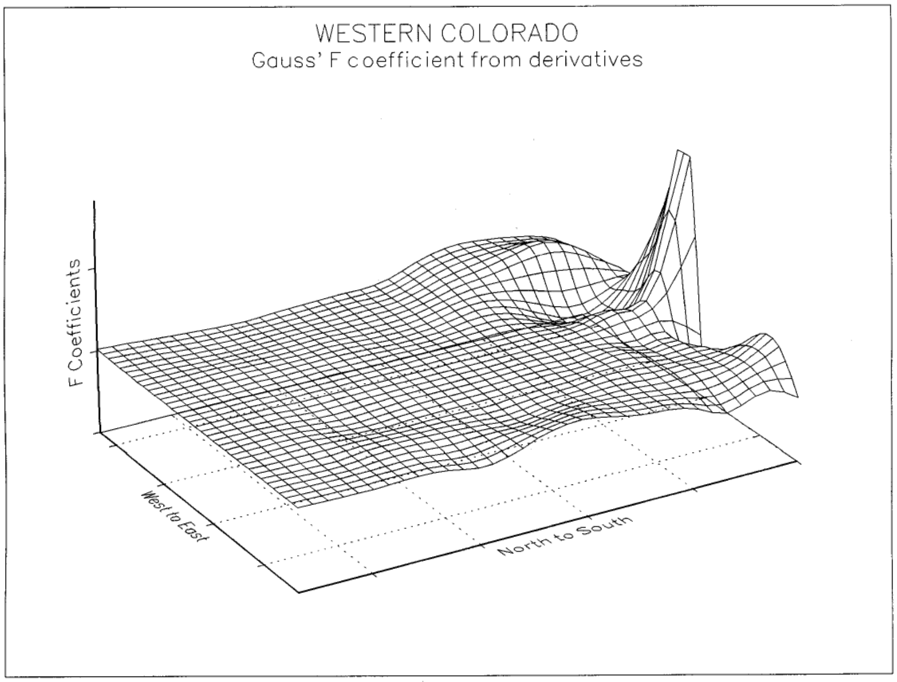
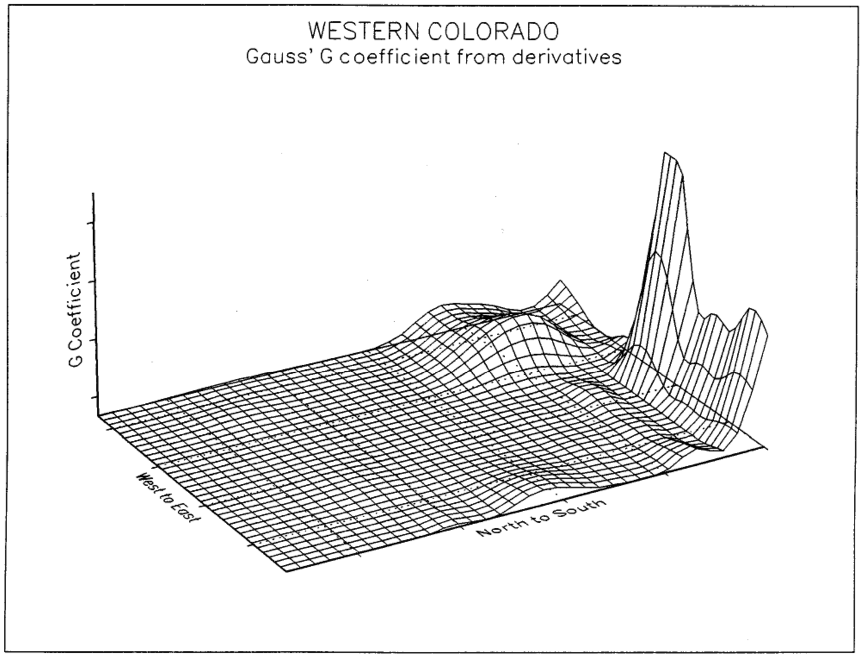


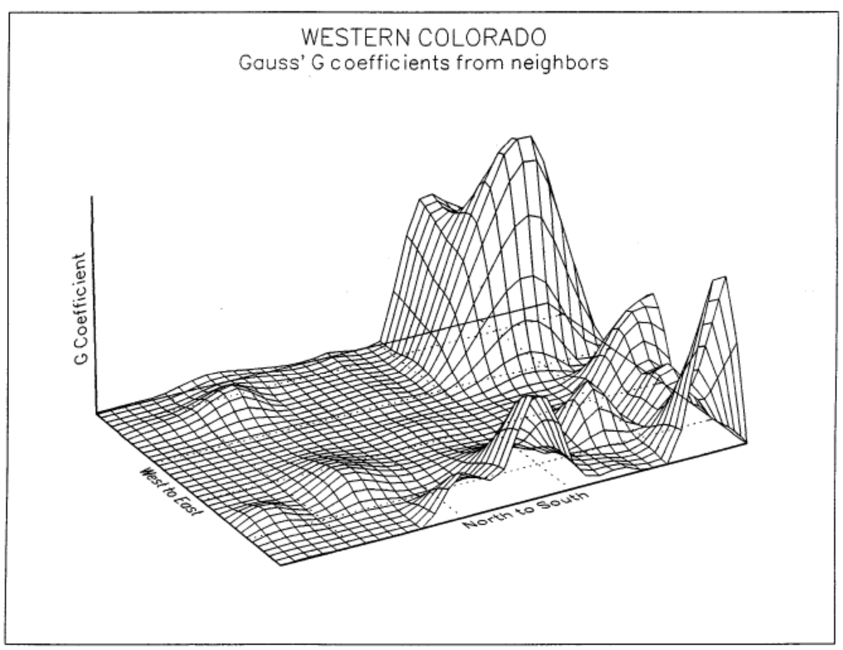
Figure 4.
By interpolation from the data represented in Figure 3 one can construct a transportation "surface", with the ratios between the planar and surface distances between adjacent locations yielding the Gaussian coefficients. See text.


WESTERN COLORADO
Gauss' E coefficients from derivatives
E Coefficients
West to East
North to South

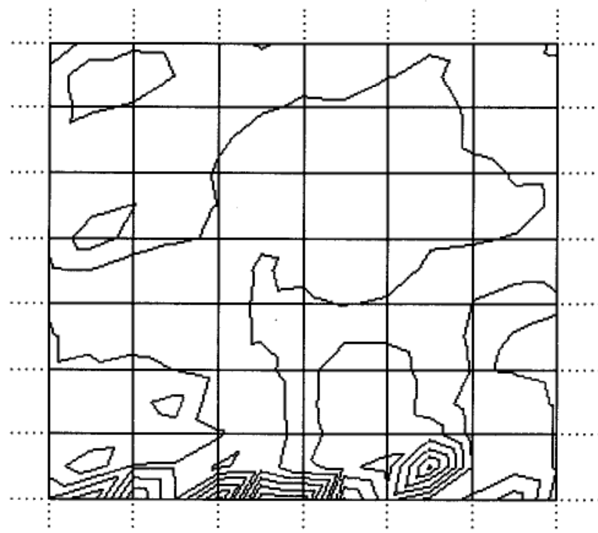

WESTERN COLORADO
Gauss' E coefficients from neighbors
E Coefficients
West to East
North to South


WESTERN COLORADO
Gauss' F coefficient from derivatives
F Coefficients
West to East
North to South


WESTERN COLORADO
Gauss' G coefficient from derivatives
G Coefficient
West to East
North to South


WESTERN COLORADO
Gauss' G coefficients from neighbors
G Coefficient
West to East
North to South

curvature
The contour interval is .1210559 The array is 15 by 13 Grid interval is 2



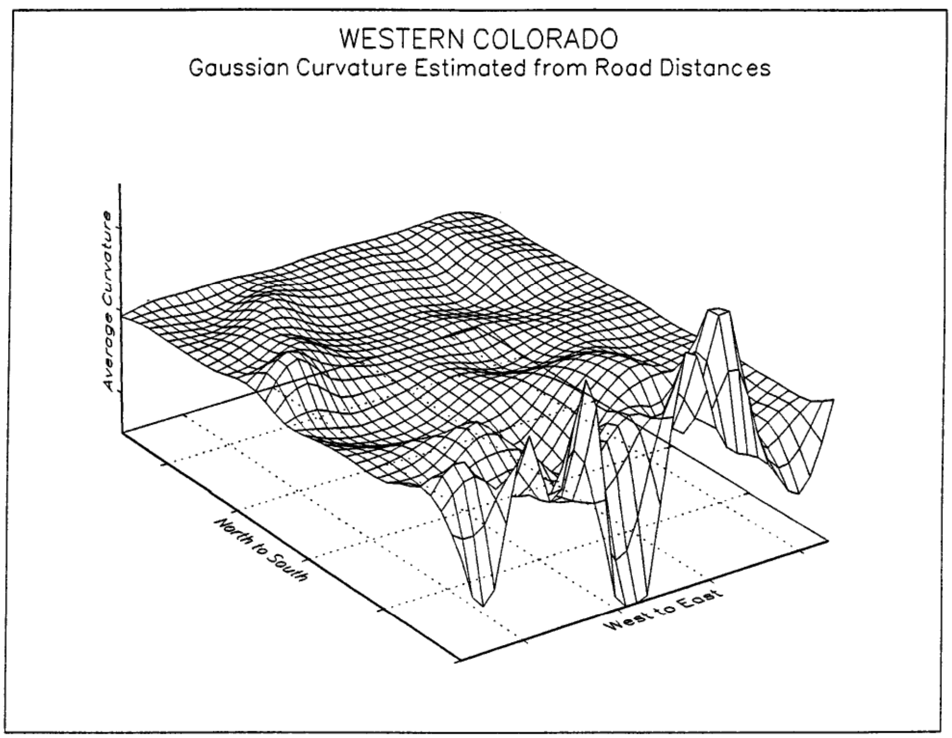


Figure 5.
Gaussian curvature calculated from road distances in western Colorado, using the data represented in the previous figure.

Actual versus road distance coordinates

X	Y	U	V
90.25032	-158.1648	143.3976	-173.4812
6.341792	32.86065	.6107957	71.51968
144.3592	-48.71337	199.5391	-31.10237
117.8317	101.6162	194.5898	166.7464
-149.631	-170.652	-181.2717	-311.1748
-55.11283	180.6381	-80.91472	264.778
-87.35231	-179.7049	-115.0802	-273.1995
116.2979	165.7561	224.1675	268.5189
152.8661	189.034	271.5455	239.8979
102.3003	90.82483	145.641	149.4794
-36.86908	72.25089	-55.00954	102.1502
-143.2056	20.62821	-177.4204	18.9895
-3.170864	-38.70581	11.37794	-102.2559
51.70794	39.51811	91.44124	73.76418
-87.29237	127.9107	-133.5379	187.5752
-85.61292	-45.66383	-132.2457	-89.91963
-68.34016	-96.63337	-162.7283	-143.0159
-11.05129	-181.4572	24.05249	-277.2025
77.61503	-40.18304	96.20037	-37.17067
-68.10146	-120.1828	-160.4169	-189.8122
5.097425	177.1015	7.205933	259.8088
-80.56223	-106.3481	-229.2311	-156.6576

Colorado X, Y, Z

90.2503	-158.16	0.00000
6.34179	32.8607	1.84969
144.359	-48.713	2.58861
117.832	101.616	2.02529
-149.63	-170.65	2.10327
-55.113	180.638	2.41604
-87.352	-179.70	1.91409
116.298	165.756	1.76447
152.866	189.034	2.15203
102.300	90.8248	2.14048
-36.869	72.2509	2.04427
-143.21	20.6282	2.37293
-3.1709	-38.706	1.69537
51.7079	39.5181	1.87494
-87.292	127.911	2.58006
-85.613	-45.664	1.79898
-68.340	-96.633	1.79954
-11.051	-181.46	2.54959
77.6150	-40.183	2.10000
-68.101	-120.18	1.53017
5.09743	177.102	2.02796
-80.562	-106.35	1.35885

Colorado slant height array 15 by 13

1.930	2.169	2.373	2.467	2.418	2.235	2.030	1.952	1.882	1.807	1.707	2.030	2.288
1.934	2.172	2.374	2.467	2.409	2.234	2.042	1.954	1.883	1.807	1.707	2.010	2.244
1.977	2.215	2.485	2.618	2.467	2.250	2.064	1.956	1.887	1.831	1.802	2.060	2.296
1.991	2.228	2.492	2.597	2.408	2.186	2.032	1.948	1.921	1.934	1.951	2.146	2.382
1.980	2.218	2.393	2.401	2.220	2.038	1.951	1.914	1.948	2.075	2.174	2.301	2.536
2.039	2.277	2.347	2.257	2.068	1.922	1.883	1.858	1.900	2.076	2.251	2.432	2.668
2.340	2.506	2.373	2.179	1.990	1.871	1.851	1.825	1.878	2.088	2.302	2.511	2.747
2.092	2.284	2.201	2.044	1.898	1.791	1.762	1.814	1.947	2.162	2.365	2.567	2.802
1.623	1.860	1.916	1.874	1.820	1.725	1.682	1.801	1.993	2.217	2.396	2.591	2.828
1.380	1.616	1.729	1.788	1.834	1.754	1.689	1.775	1.922	2.099	2.272	2.534	2.795
1.259	1.495	1.635	1.789	1.906	1.842	1.757	1.698	1.658	1.675	1.815	2.037	2.272
1.233	1.469	1.495	1.465	1.695	1.831	1.782	1.558	1.271	1.064	1.211	1.514	1.749
1.417	1.652	1.609	1.549	1.692	1.959	1.905	1.459	.897	.370	.605	1.045	1.280
2.046	2.114	1.886	1.870	1.963	2.328	2.230	1.480	.747	-.138	.245	.768	1.004
2.094	2.202	1.908	1.942	1.963	2.669	2.630	1.480	.747	-.138	.245	.768	1.004

Gaussian curvatures from derivatives 15 by 13

-0.028024	0.001430	0.001467	0.004998	0.000493	-0.010119	-0.001470	0.005470	0.000995	0.000496	0.009877	0.020280	-0.018484
0.000473	-0.016865	-0.037311	-0.007009	0.025795	0.003348	0.003716	0.007204	-0.004988	-0.022895	0.036329	0.015779	0.010429
0.000473	-0.015216	-0.004073	0.078261	0.050512	-0.000221	-0.014400	-0.012294	-0.032720	-0.049080	0.012887	0.024125	-0.000943
0.000000	0.021763	0.069347	0.088995	0.002844	-0.037467	-0.045147	-0.044824	-0.071710	-0.076443	0.013312	0.033770	0.000467
-0.000473	0.042083	0.087210	0.037556	-0.017927	-0.043177	-0.042451	-0.033494	-0.064550	-0.074994	-0.038722	0.000862	0.000000
0.033056	0.063612	0.115957	0.044550	-0.010149	-0.025154	-0.018929	-0.005947	-0.021731	-0.048030	-0.047699	-0.022390	-0.000469
0.022368	0.184547	0.064994	0.005453	-0.020835	-0.014984	-0.021319	-0.040235	-0.017894	-0.015764	-0.011746	-0.005333	0.000472
-0.030462	-0.008534	-0.072086	-0.068949	-0.038314	0.000834	-0.029556	-0.070548	-0.038832	0.005228	0.011773	0.002091	-0.000473
-0.018598	-0.081711	-0.106658	-0.101118	-0.058140	0.007153	0.014554	-0.012639	0.002162	0.023795	-0.009623	-0.023786	-0.012309
0.000459	-0.033729	-0.073040	-0.092459	-0.057685	0.003604	0.064114	0.097802	0.098606	0.056812	-0.023537	-0.005074	0.000873
0.000000	0.012136	0.041878	-0.011226	-0.048509	-0.055530	0.071854	0.182211	0.178548	0.062403	-0.012026	-0.003115	0.009784
0.000471	0.002062	0.085011	0.149434	-0.104123	-0.115601	0.070159	0.208110	0.221880	0.058085	-0.055456	-0.044124	0.000000
0.068826	0.025096	0.054982	0.075629	0.010355	-0.114026	0.058028	0.182456	0.211585	0.182444	-0.047187	-0.075505	-0.000419
0.056737	0.189648	-0.014278	-0.019365	-0.142848	-0.087893	0.094075	0.140905	0.065930	0.617857	-0.005939	-0.086399	-0.000465
-0.136772	-0.343508	0.352959	-0.016160	0.522867	-0.592702	-0.574193	0.488044	0.162184	-0.046344	-0.157513	-0.317595	0.073411

Gaussian E coefficients from derivatives 15 by 13

1.057121	1.049062	1.022201	1.000506	1.013456	1.037636	1.020022	1.005476	1.005256	1.007656	1.012432	1.084390	1.066564
1.056644	1.048400	1.021756	1.000306	1.013572	1.033672	1.019600	1.006320	1.005402	1.007744	1.010302	1.072092	1.054756
1.056644	1.064516	1.040602	1.000081	1.033856	1.040602	1.021609	1.007832	1.003906	1.001806	1.013110	1.061009	1.055696
1.056169	1.062750	1.034040	1.001764	1.042230	1.035344	1.014161	1.003080	1.000049	1.000225	1.011236	1.046440	1.055696
1.056644	1.042642	1.008372	1.007482	1.032942	1.018090	1.003844	1.000002	1.006480	1.012769	1.012769	1.032761	1.055225
1.056644	1.023716	1.000100	1.019460	1.028056	1.008556	1.001024	1.000072	1.011881	1.030800	1.031684	1.043472	1.055696
1.027556	1.000272	1.026732	1.036672	1.023716	1.004830	1.000529	1.000182	1.017292	1.044944	1.044732	1.049506	1.055696
1.036864	1.002970	1.014400	1.022952	1.016002	1.004624	1.000132	1.008556	1.030276	1.043681	1.041006	1.047742	1.055225
1.056169	1.021462	1.000049	1.002304	1.005550	1.004761	1.001444	1.024180	1.043264	1.040602	1.034969	1.046656	1.056169
1.055696	1.030450	1.007396	1.002756	1.000289	1.005256	1.000110	1.013572	1.026244	1.030625	1.047306	1.068382	1.068121
1.055696	1.035344	1.021609	1.018360	1.000702	1.005550	1.005184	1.002450	1.000132	1.006162	1.032761	1.052212	1.055225
1.055696	1.017161	1.000004	1.010000	1.033489	1.001892	1.018632	1.065280	1.061009	1.000900	1.050625	1.072361	1.055225
1.055225	1.009216	1.002652	1.001722	1.042025	1.011342	1.062500	1.254016	1.296480	1.021316	1.113906	1.113906	1.055225
1.004624	1.006400	1.014884	1.001482	1.052441	1.017822	1.179776	1.549822	1.654481	1.063001	1.205209	1.144020	1.055696
1.011664	1.008649	1.016900	1.000756	1.132132	1.111222	1.353430	1.886422	1.654481	1.063001	1.205209	1.144020	1.055696

Gaussian F coefficients from derivatives 15 by 13

-0.001912	-0.001329	-0.000298	0.000000	-0.002088	-0.000388	0.003396	0.000296	0.000145	0.000000	0.000000	0.011620	0.022704
-0.011186	-0.010120	-0.016520	-0.002642	0.005709	0.002753	0.004760	0.000318	0.000367	0.002112	-0.009643	-0.008055	-0.001872
-0.013566	-0.014224	-0.023777	0.001170	-0.000184	-0.009672	-0.001470	-0.000531	0.002375	0.005397	-0.027938	-0.033592	-0.032568
-0.000711	-0.000752	0.016974	-0.009114	-0.050758	-0.039856	-0.013447	-0.002331	0.000427	-0.003660	-0.039432	-0.051936	-0.056640
-0.011424	-0.010119	0.013268	-0.029410	-0.061710	-0.035508	-0.009238	-0.000135	0.001691	-0.016046	-0.033900	-0.051766	-0.067210
-0.085680	-0.044352	-0.000200	-0.030969	-0.038525	-0.015448	-0.003200	0.000756	0.007630	-0.002282	-0.022784	-0.043785	-0.049796
-0.008798	-0.000115	-0.023871	-0.040789	-0.026180	-0.009105	-0.002783	0.000594	-0.006181	-0.018232	-0.024111	-0.030037	-0.031624
0.137664	0.035207	-0.054840	-0.046207	-0.021505	-0.009928	0.001943	0.002220	-0.020010	-0.026961	-0.019035	-0.017480	-0.019035
0.168744	0.097862	0.003304	-0.012288	-0.004768	-0.002553	0.002774	0.006065	0.005200	0.012694	0.017391	0.007128	0.001659
0.085904	0.063692	0.024166	0.004462	0.001462	0.008482	-0.000788	0.012000	0.054270	0.094850	0.126367	0.144871	0.145116
0.034692	0.027636	0.034398	0.043766	0.003683	0.005737	0.006696	-0.010741	-0.007487	0.081248	0.192041	0.233070	0.245810
-0.037288	-0.020567	-0.000052	0.024000	0.039162	-0.005089	0.020202	-0.061064	-0.187967	-0.039150	0.272250	0.266848	0.233120
-0.191055	-0.061920	0.020136	-0.016808	-0.054940	-0.052930	0.112000	-0.039312	-0.285318	-0.175492	0.326025	0.251775	0.175075
-0.046036	0.044000	0.036478	-0.015131	-0.062059	-0.094785	0.307400	0.015572	-0.121350	-0.127508	0.163080	0.105121	0.065136
-0.231336	0.212970	0.250900	-0.055385	-0.713550	-1.003835	1.801335	1.393420	0.604323	-0.034638	-0.110985	-0.291456	-0.236944

Gaussian G coefficients from derivatives 15 by 13

1.000016	1.000009	1.000001	1.000000	1.000081	1.000001	1.000144	1.000004	1.000001	1.000000	1.000000	1.000400	1.001936
1.000552	1.000529	1.003136	1.005700	1.000600	1.000056	1.000289	1.000004	1.000006	1.000144	1.002256	1.000225	1.000016
1.000812	1.000784	1.003481	1.004225	1.000000	1.000576	1.000025	1.000009	1.000361	1.004032	1.014884	1.004624	1.004761
1.000002	1.000002	1.002116	1.011772	1.015252	1.011236	1.003192	1.000441	1.000930	1.014884	1.034596	1.014520	1.014400
1.000576	1.000600	1.005256	1.028900	1.028900	1.017424	1.005550	1.002025	1.000110	1.005041	1.022500	1.020449	1.020449
1.032400	1.020736	1.000100	1.012321	1.013225	1.006972	1.002500	1.001980	1.001225	1.000042	1.004096	1.011025	1.011130
1.000702	1.000012	1.005329	1.011342	1.007225	1.004290	1.003660	1.000484	1.000552	1.001849	1.003249	1.004556	1.004489
1.128522	1.104329	1.052212	1.023256	1.007225	1.005329	1.007140	1.000144	1.003306	1.004160	1.002209	1.001600	1.001640
1.126736	1.111556	1.055696	1.016384	1.001024	1.000342	1.001332	1.000380	1.000156	1.000992	1.002162	1.000272	1.000012
1.033124	1.033306	1.019740	1.001806	1.001849	1.003422	1.001406	1.002652	1.028056	1.073441	1.084390	1.076729	1.077284
1.005402	1.005402	1.013689	1.026082	1.004830	1.001482	1.002162	1.011772	1.105950	1.267806	1.281430	1.260100	1.273529
1.006241	1.006162	1.000169	1.014400	1.011449	1.003422	1.005476	1.014280	1.144780	1.425756	1.366025	1.246016	1.246016
1.165242	1.104006	1.038220	1.041006	1.017956	1.061752	1.050176	1.001521	1.068644	1.361201	1.233289	1.139129	1.138756
1.114582	1.075625	1.022350	1.038612	1.018360	1.126025	1.131406	1.000110	1.005625	1.064516	1.032400	1.019182	1.019044
2.147041	2.311025	1.931225	2.014049	1.963342	3.265025	3.295225	1.547600	1.139502	1.004761	1.015006	1.147456	1.252004

E coefficients from neighbors 15 by 12

1.028164	1.020596	1.004408	1.001200	1.016607	1.020796	1.003037	1.002447	1.002809	1.004988	1.050871	1.032746
1.027932	1.020198	1.004315	1.001681	1.015197	1.018265	1.003865	1.002517	1.002884	1.004988	1.044897	1.027013
1.027932	1.035809	1.008806	1.011336	1.023274	1.017151	1.005815	1.002378	1.001567	1.000420	1.032746	1.027471
1.027701	1.034261	1.005497	1.017704	1.024346	1.011788	1.003522	1.000364	1.000085	1.000144	1.018835	1.027471
1.027932	1.015197	1.000032	1.016248	1.016427	1.003777	1.000684	1.000578	1.008032	1.004889	1.008032	1.027241
1.027932	1.002447	1.004042	1.017704	1.010602	1.000760	1.000312	1.000882	1.015370	1.015197	1.016248	1.027471
1.013684	1.008806	1.018644	1.017704	1.007056	1.000200	1.000338	1.001404	1.021812	1.022642	1.021607	1.027471
1.018265	1.003439	1.012249	1.010602	1.005708	1.000420	1.001351	1.008806	1.022851	1.020396	1.020198	1.027241
1.027701	1.001567	1.000882	1.001457	1.004502	1.000924	1.007056	1.018265	1.024781	1.015894	1.018835	1.027701
1.027471	1.006364	1.001739	1.001057	1.003195	1.002110	1.003691	1.010747	1.015544	1.014854	1.033752	1.033499
1.027471	1.009752	1.011788	1.006821	1.002046	1.003606	1.001739	1.000800	1.000144	1.009753	1.024346	1.027241
1.027471	1.000338	1.000450	1.026109	1.009206	1.001200	1.024781	1.040370	1.021200	1.010747	1.044897	1.027241
1.027241	1.000924	1.001798	1.010173	1.035031	1.001457	1.094950	1.147102	1.130367	1.027241	1.092520	1.027241
1.002309	1.025663	1.000128	1.004315	1.064530	1.004791	1.250000	1.239875	1.335374	1.070836	1.128507	1.027471
1.005815	1.042322	1.000578	1.000220	1.224106	1.000760	1.523975	1.239875	1.335374	1.070836	1.128507	1.027471

G coefficients from neighbors 14 by 13

1.000008	1.000005	1.000000	1.000000	1.000041	1.000000	1.000072	1.000002	1.000000	1.000000	1.000000	1.000200	1.000968
1.000924	1.000924	1.006142	1.011336	1.001681	1.000128	1.000242	1.000002	1.000008	1.000288	1.004502	1.001249	1.001351
1.000098	1.000085	1.000025	1.000221	1.001739	1.002046	1.000512	1.000032	1.000578	1.005291	1.011039	1.003691	1.003691
1.000060	1.000050	1.004889	1.019027	1.017519	1.010893	1.003275	1.000578	1.000364	1.009892	1.024563	1.011941	1.011788
1.001739	1.001739	1.001057	1.010315	1.011486	1.006706	1.002309	1.001567	1.001151	1.000000	1.002960	1.008544	1.008674
1.044318	1.025885	1.000338	1.003037	1.003037	1.001300	1.000512	1.000544	1.000242	1.000072	1.001300	1.003116	1.003116
1.030293	1.024346	1.014684	1.009071	1.004223	1.003195	1.003953	1.000060	1.002378	1.002734	1.001983	1.001567	1.001511
1.104518	1.086175	1.039820	1.014347	1.003037	1.002176	1.003195	1.000085	1.001057	1.001511	1.000480	1.000288	1.000338
1.029101	1.029338	1.017334	1.003691	1.000098	1.000420	1.000025	1.000338	1.002517	1.006938	1.007659	1.001623	1.000544
1.007294	1.007294	1.004408	1.000000	1.002589	1.003865	1.002309	1.002960	1.034261	1.086175	1.099477	1.116696	1.128507
1.000338	1.000338	1.009752	1.051178	1.022018	1.000060	1.000312	1.009752	1.072273	1.171888	1.168253	1.128507	1.128507
1.016787	1.016607	1.006477	1.003522	1.000005	1.008159	1.007536	1.004889	1.067650	1.217225	1.169289	1.128507	1.128507
1.181373	1.101564	1.037656	1.050258	1.036070	1.065909	1.051487	1.000220	1.011187	1.121634	1.062826	1.037655	1.037389
1.001151	1.003865	1.000242	1.002589	1.000000	1.056542	1.077033	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000

TWENTY TWO TOWNS IN WESTERN COLORADO

Location, and converted to plane coordinates in kilometers.

	Latitude	Longitude	X	Y	Elev(m)	Name
1	37.46833	-105.8700	90.25032	-158.1648	2299	Alamosa
2	39.19000	-106.8183	6.341792	32.86065	2405	Aspen
3	38.44500	-105.2350	144.3592	-48.71337	1706	Canon City
4	39.80000	-105.5133	117.8317	101.6162	2594	Central City
5	37.34833	-108.5833	-149.6310	-170.6520	1888	Cortez
6	40.51667	-107.5433	-55.11283	180.6381	1889	Craig
7	37.27500	-107.8783	-87.35231	-179.7049	1992	Durango
8	40.37667	-105.5200	116.2979	165.7561	2298	Estes Park
9	40.58000	-105.0833	152.8661	189.0340	1525	Fort Collins
10	39.70500	-105.6967	102.3003	90.82483	2596	Georgetown
11	39.54333	-107.3217	-36.86908	72.25089	1758	Glenwood Springs
12	39.06833	-108.5500	-143.2056	20.62821	1400	Grand Junction
13	38.54667	-106.9283	-3.170864	-38.70581	2348	Gunnison
14	39.24833	-106.2917	51.70794	39.51811	3097	Leadville
15	40.04000	-107.9167	-87.29237	127.9107	1902	Meeker
16	38.48000	-107.8750	-85.61292	-45.66383	1770	Montrose
17	38.02333	-107.6717	-68.34016	-96.63337	2378	Ouray
18	37.26333	-107.0167	-11.05129	-181.4572	2168	Pagosa Springs
19	38.53000	-106.0000	77.61503	-40.18304	2155	Salida
20	37.81167	-107.6667	-68.10146	-120.1828	2837	Silverton
21	40.48667	-106.8317	5.097425	177.1015	2053	Steamboat Springs
22	37.93500	-107.8100	-80.56223	-106.3481	2680	Telluride

Conversion from latitude and longitude to kilometers on a local map projection (Tobler, 1974), using a radius of 6373.592 km based on the mean radius of the Clarke ellipsoid of 1866 at 38.89N, 106.89W.

[1] In the present instance the Colorado data were converted from latitude and longitude coordinates using a local map projection (Tobler, 1974) for which the spherical distances did not differ from the map distance by more than 1/3 kilometer.

References

Angel, S. & Hyman, G., 1976, *Urban Fields*, Pion:London
Beckmann, M., 1952, "A Continuous Model of Transportation", *Econometrica*, 20:643-660
Blumenthal, L, 1953, *Theory and Applications of Distance Geometry*, Oxford, Clarendon.
Boggs, S., 1941, “Mapping the Changing World”, *Annals*, AAG XXXI, 2: 119-128.
Dorigo, G., & Tobler, W., 1983, "Push-Pull Migration Laws", *Annals*, AAG, 73(1):1-17
Eisenhart, L., 1960, *A Treatise on the Differential Geometry of Curves and Surfaces*, New York, Dover.
Gatrell, A., 1983, *Distance and Space*, Clarendon Press: Oxford
Gauss, C., 1902, *General Investigation on Curved Surfaces (1827)*, Princeton U.P. (Dover reprint 2005)
Iri, M., 1980, "Theory of Flows in Continua as Approximation to Flows in Networks", in A. Prekopa, ed., *Survey of Mathematical Programming, 2*, North Holland: Amsterdam
Kreyszig, E., 1959, *Differential Geometry*, Toronto U.P.
Lane, E., 1940, *Metric Differential Geometry of Curves and Surfaces*, Chicago U.P.
Lősch, A., 1954, *The Economics of Location*, New Haven, Yale U.P
Love, R., & Morris, J. 1972, "Mathematical Models of Road Travel Distances", *Management Science*, 25:130-139
Love, R., & Morris, J., 1979, "Modeling Inter-City Road Distances by Mathematical Functions", *Operational Research Quarterly*, 23:61-71
Mayhew, L., 1986, *Urban Hospital Location*, Allen & Unwin: London
Misner, C., C., Thorne, K., & Wheeler, J., 1973, *Gravitation*, Freeman: San Francisco.
Morse, M., 1960, “Fields of Geodesics Issuing from a Point”, *Proceedings*, 105-111, NAS
Muller, J-C, 1978, "The Mapping of Travel Time in Edmonton, Alberta", *The Canadian Geographer*, 22(3):195-210
Muller, J-C, 1984, "Canada's Elastic Space: A Portrayal of Route and Cost Distances", The Canadian Geographer, 18:46-62
Nicholson, T., 1966, "Finding the shortest route between two points in a network", *Computer Journal*, 9:275-280
O'Neill, B., 1966, *Elementary Differential Geometry*, Academic Press: New York
Rektorys, K., 1969, *A Survey of Applicable Mathematics*, Cambridge, MIT.
Riedel, J., 1911, “Neue Studien űber Isochronenkarten”, *Petermann’s Geog. Mitt.*, LII, 281-284
Riemann, B., 1854, “Űber die Hypothesen welche der Geometrie zugrunde liegen”, Dissetation, (Reprinted in Weyl, op. cit.)
Schilling, F., 1928, "Konstruction kuerzester Wege in einem Gelaende", *Zeitschrift fuer Angewandte Mathematik and Mechanik*, 8:45-68
Spivak, M., 1970, *A Comprehensive Introduction to Differential Geometry*, Five Vols., Boston, Publish or Perish Press.
Stewart, J., & W. Warntz, 1958, “Macrogeography and Social Science” *The Geographical Review*, XVIII, 167-184.
Struik, D., 1950, *Lectures on Classical Differential Geometry*, Reading, Addison-Wesley
Taguchi, A., & Iri, M., 1982, "Continuum Approximation to Dense Networked and its Application to the Analysis of Urban Road Networks", *Mathematical Programming Study*, 20:178-217
Thompson, D’Arcy, 1948, *On Growth and Form*, New York, Macmillan.

Thűnen, J. 1826, *Der isolierte Staat in Beziehung auf landwirtschaft und Nationalokonomike*, Jena, Gustav Fischer.

Tissot, M. 1881, *Memoire sur la Representation des Surfaces et les Projections des Cartes Geographiques,* Gautier-Villars: Paris

Tobler, W, 1974, "Local Map Projections", *The American Cartographer,* 1:51-62

Tobler, W., 1978, Migration Fields, in W. Clark & E. Moore, eds., *Population Mobility and Residential Change*, Studies in Geography No. 25, Department of Geography: Evanston

Tobler, W., 1993, "Non-Isotropic Modeling" and "Speculations on the Geometry of Geography", in *Three Presentations of Geographical Analysis and Modeling*, Technical Report 93-1, National Center for Geographic Information and Analysis: Santa Barbara

Tobler, W., 1994, "Bidimensional Regression", *Geographical Analysis,* 26(3):186-212

Trunin, Y., & Serbenyuk, S., 1968, "Maps of Accessibility in Analysis of Economic-Geographic Space: Cartographic Transformation of a Surface of Negative Curvature" (in Russian), *Voprosii Geographii*, 179-186

Vadnal, A., 1958, “La localization du Reseau de Routes Collectrices dans la Circle”, *Congress European d’Econometrie*, Bilboa.

Vaughan, R., 1987, *Urban Spatial Traffic Patterns,* Pion: London

Weyl, H., 1923, “Matematische Analsyse des Raumproblems”, Barcalona Lectures, Springer, Berlin (Reprinted in H. Weyl, Das Kontinium und andere Monographien, New York, Chelsea)

Wingo, L., 1961, *Transportation and Urban land,* Resources for the Future, Washington D.C.

Zaustinsky, E., 1959, *Spaces with Non-Symmetric Distance*, *Memoir 34,* American Mathematical Society.

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Part Two

Abstract
Empirical measurements of spatial separations before and after transportation innovations can be considered to be spatial transformations and, as such, evaluated using Tissot's strain tensor. These graphical quantities should be of practical benefit in evaluating transportation systems. Examples using empirical road distances are given.

We all know that modern transportation systems change spatial relations between places. One approach is to consider that the transportation induces a distortion, warping, or transformation of the two dimensional (spherical, or locally Euclidean) space. Felix Klein, a famous German mathematician defined geometry as the study of properties preserved by a particular group of transformations. Thus one has projective transformations and projective geometry, affine transformations and affine geometry, Euclidean transformations and Euclidean geometry, continuous transformations and topology, etc., The method of transformations has proven highly fruitful in many fields. Poincaré, for example, mapped non-Euclidean geometry onto a plane, thereby proving the consistency of this non-Euclidean geometry. Straight line in this geometry, incidentally, were to be drawn on paper using a compass, i.e., they are arcs of circles. In biology D'Arcy Thompson has applied continuous transformation to illustrate relations among species. Dűrer used transformations in a number of sketches. In aerodynamics transformations are employed to study the flow of wind past a wing by mapping the wing shape into the shape of a cylinder, solving the flow problem, and then transforming the resulting solution into one that applies to the more complicated wing shape. Transformations are familiar to geographers from the subject of map projections, which represent a special case. Areal cartograms are an example of a transformation that can be used to even-out and unequal distribution of resources.

Given a travel symmetric disutility matrix D_{ij} between places, it is possible to look at how it warps the region, and then to calculate how much distortion has taken place. The measure of distortion which I use is known in the cartographic literature as Tissot's indicatrix, and in mechanics as the strain tensor – a matrix made up of the four partial derivatives of the two dimensional mapping.

The figures illustrate these concepts. Figure XX shows the base space (assumed Euclidean'), with the displacement vectors of the towns when road distance is used in place of the spherical great circle distance between these same objects. The next figure (Figure XX) illustrates how the regular grid from the previous figure is warped by the road distances. Tissot's indicatrix is shown next (Figure XX) as the distortion of infinitesimal circles on the base, and this gives an idea of the effect of the Rocky Mountains on roads in this part of the world. In particular, the indicatrix shows the amount of stretching of distances in every direction at each location for which it is calculated. The axes of the ellipses (not shown here, but easily estimated) show the maximal and minimal stretching; they are related to the eigenvalues of the strain tensor and are based on the estimated partial derivatives of the transformation. The contour map in the next diagram (Figure XX) shows the areal distortion introduced by the mountainous terrain on the road system. Figure XX shows the maximum angular distortion (Tissot's ω). The final contour map (Figure XX) is a scalar measure of the total distortion, computed as the sum of the squares of the four partial derivatives of the transformation.

Measuring the effect of mountainous terrain on a road system is mostly of interest because the data are easily obtainable. Before and after studies of transportation innovation are likely to be more interesting but the data are more difficult to obtain. However, the techniques used to illustrate these could be the same. It is hoped that a future report will extend the present work.

Notes:

[2] The base of the arrows is at the correct planar location (see note 1) and the head of the arrow is where it would be displaced to assuming the road distances are true distances. A similarity transformation has been applied to bring the two data sets into maximal correspondence, with the center at the mean location of the data. This is an arbitrary transformation which leaves the strain tensor invariant. The results are thus independent of the coordinates used.

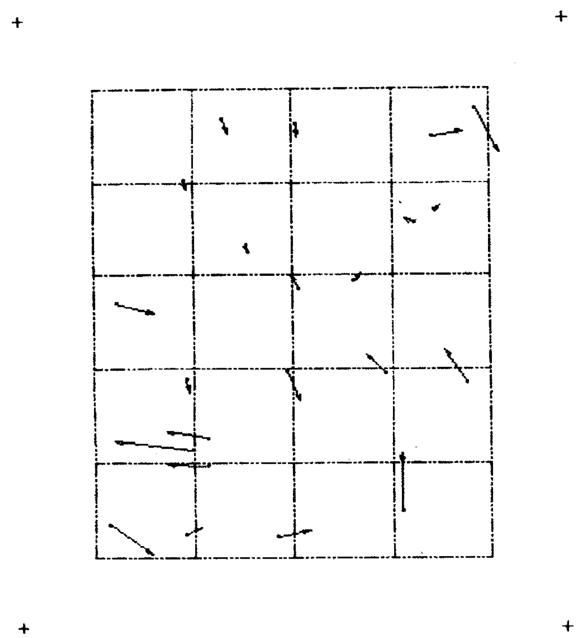

Figure 3

Colorado locations with displacements due to the difference between road distance and spherical distances.

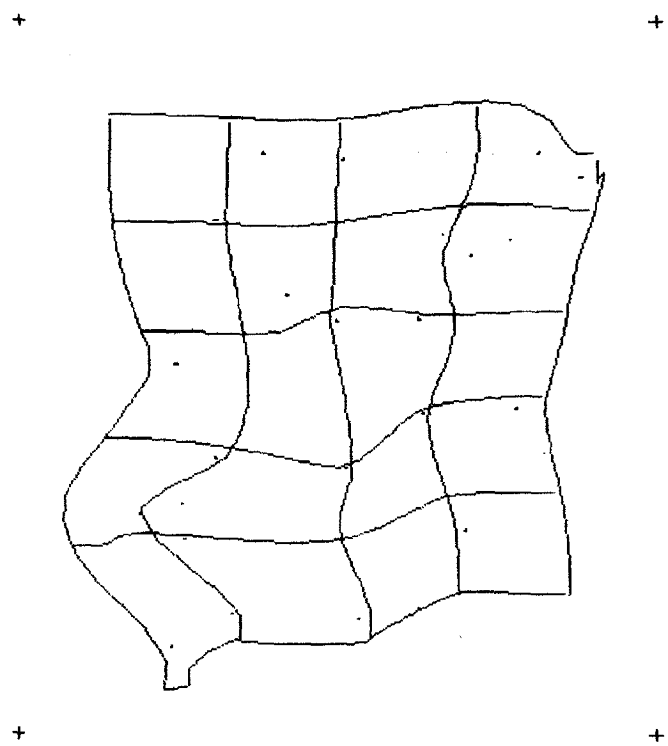

Figure 4

Distorting the grid from the previous figure to match the road distances.

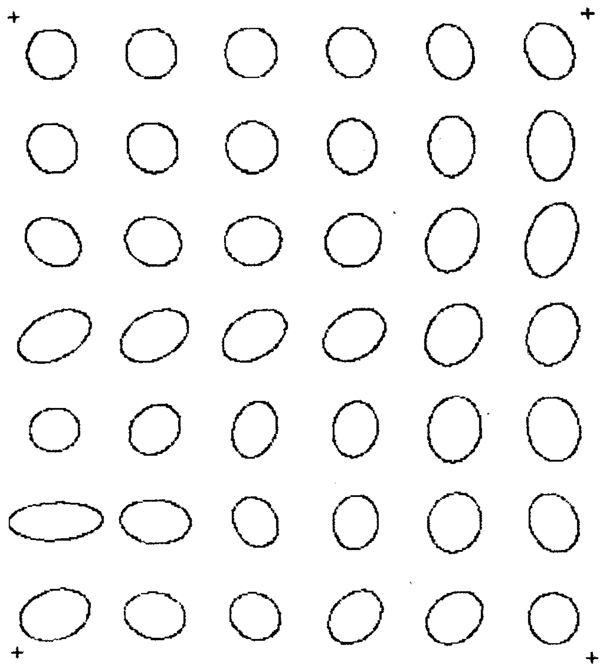

Figure 5

Tissot's indicatrix (the strain tensor) shown on the map of Western

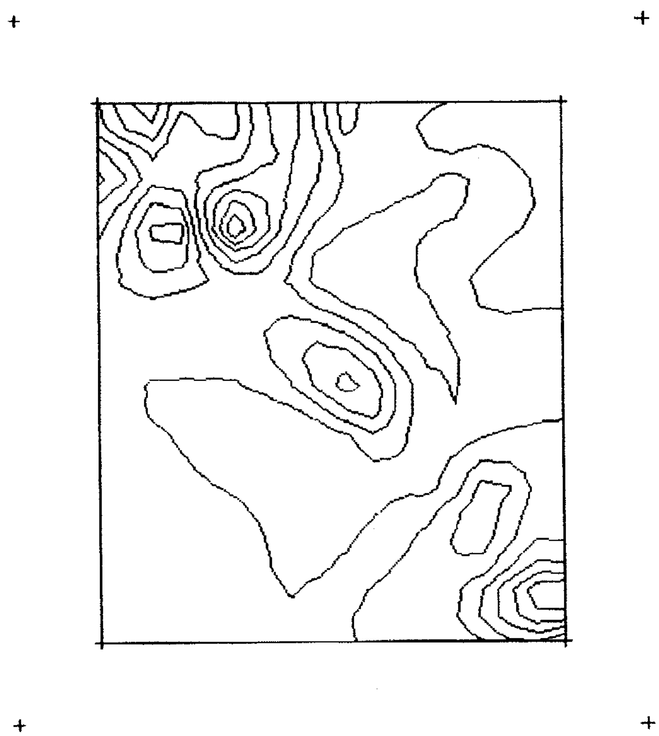

Figure 6

Areal distortion due to the mountains of Western Colorado. Tissot's "S".

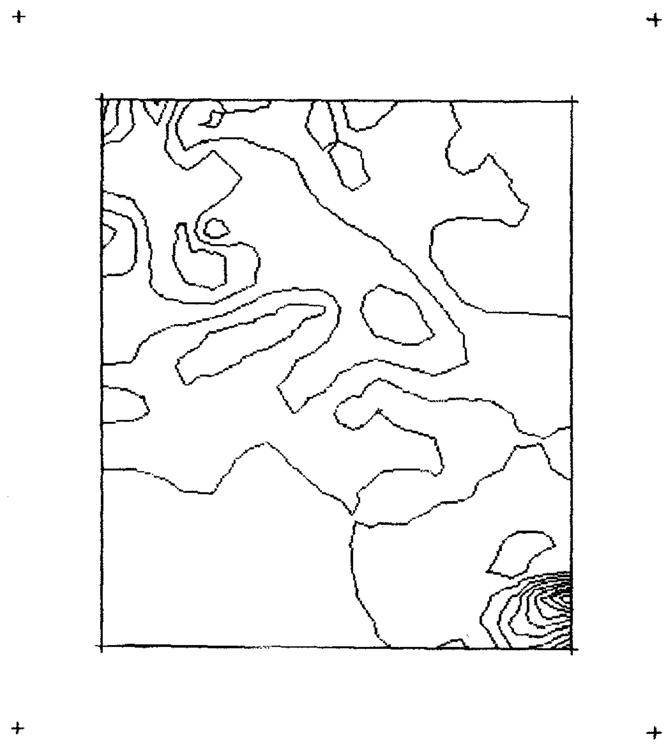

Figure 7

Angular distortion due to the mountains of Western Colorado.

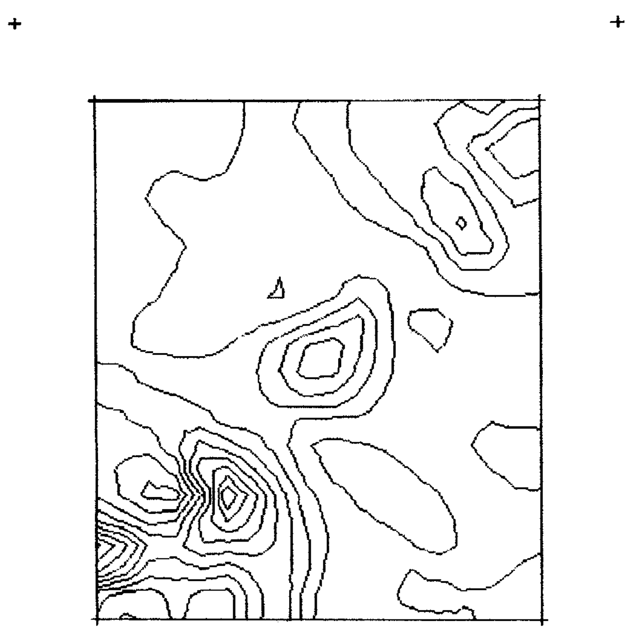

Figure 8

Total distortion, computed as the sum of squares of the partial derivatives at each point on the map of Western Colorado.