Geographical Interpolation

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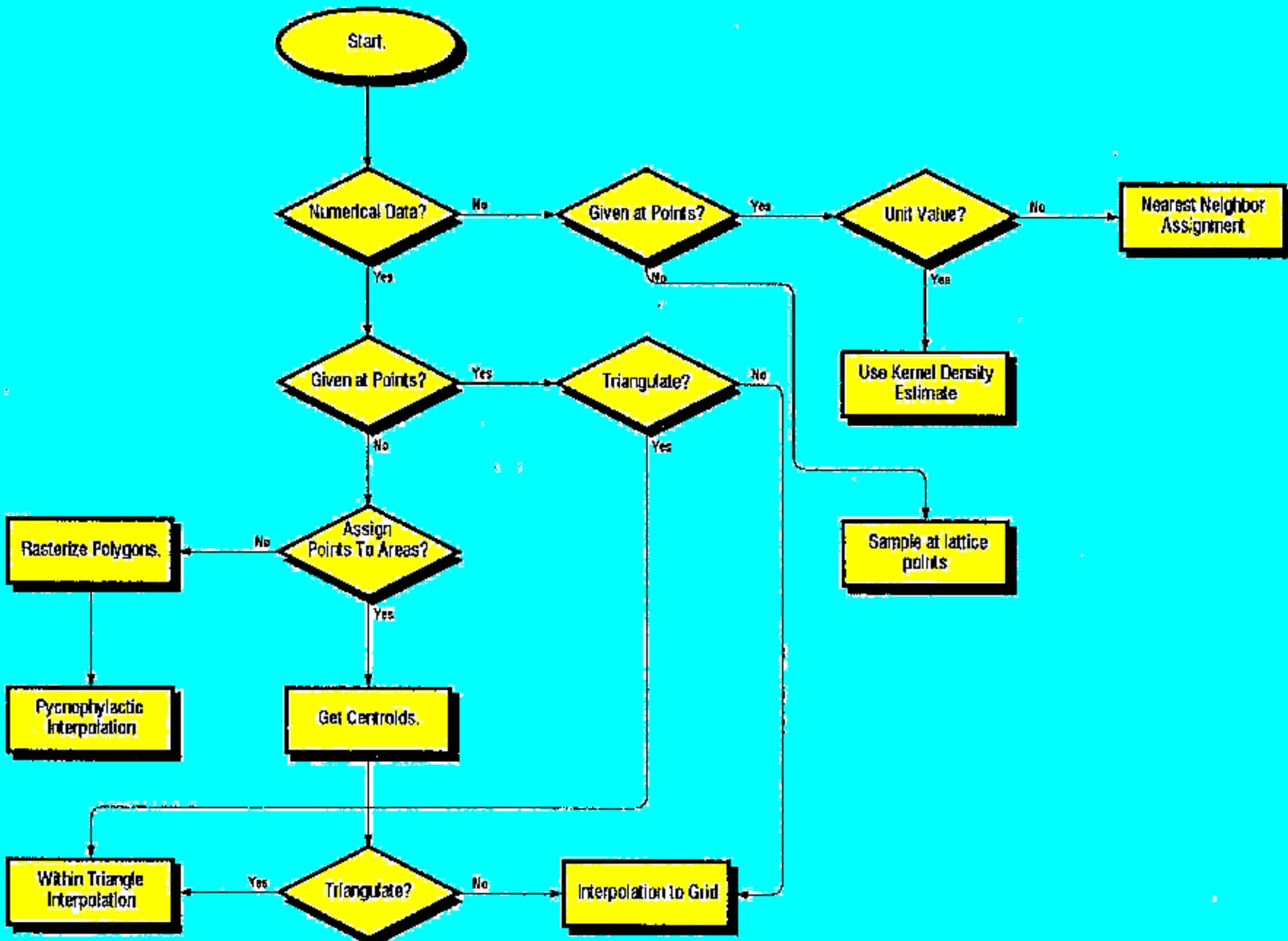
GEOGRAPHIC INTERPOLATION



Interpolation requires assumptions about the nature of the phenomena to be interpolated

Followed by “intelligent” guessing

For example

How effective is random elevation sampling, followed by interpolation, here?

Another interpolation challenge

Sample elevations at random locations in a city, then interpolate.

What do you get?

Areal Interpolation is a Special Case of Geographical Interpolation

Sometimes one has numerical observations given at point locations.

The objective is often to produce a contour map.

There is a large literature on interpolation from point data.

We mention only Kriging, inverse distance, splining, and so on.

Interpolation Criteria

- Contours based on interpolation should match the observations, unless smoothing is used to suppress observational noise or error, or map generalization is desired.
- Stated in another way the contours should be of high fidelity. In going to the field one should be able to measure the value at a point and observe the same value as shown on the contour map at that location.

But often we have observations assembled by areal units

Census tracts, school districts, and the like

Hence the need for areal interpolation.

It is incorrect, in my opinion, to assign these observations to points (centroids).

One criterion to be satisfied is that the resultant maintain the data values within each unit.

This is why I invented pycnophylactic reallocation.

Pycnophylactic Reallocation

Allows the production of density or contour maps to be made from areal data.

It is reallocation - and somewhat of a disaggregation operator. My assertion is that it may actually improve the data.

It is also important for the conversion of data from one set of statistical units to another, as from census tracts to school districts.

An example

Population Density by County

Observe the discontinuities at the county boundaries.

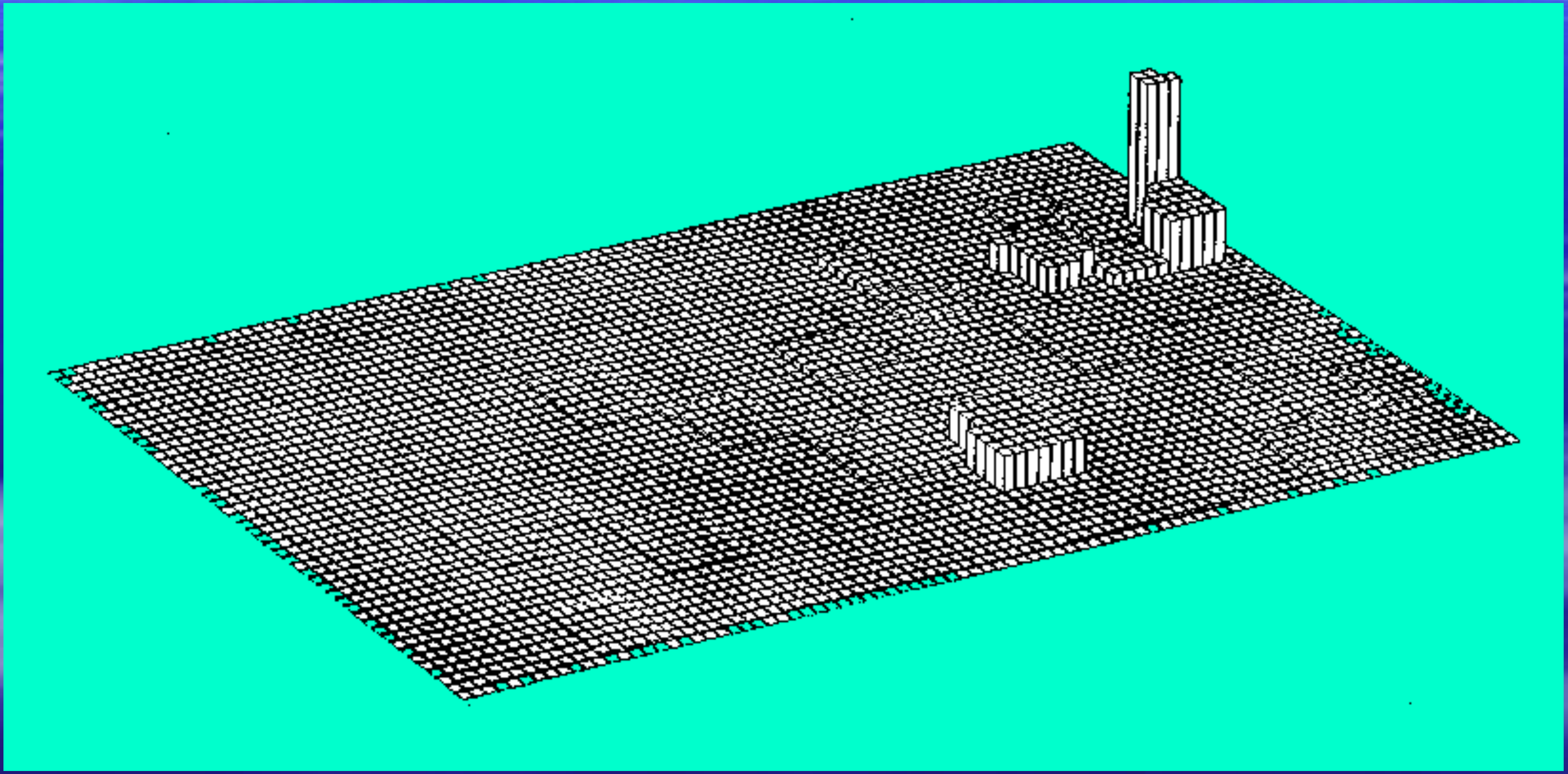
We would like a smooth map of population density, in order to draw contours.

The usual interpolation procedure will not work unless we use centroids and this fiction could allow people to be moved from one county to another.

Population Density in Kansas

By County

Courtesy of T. Slocum

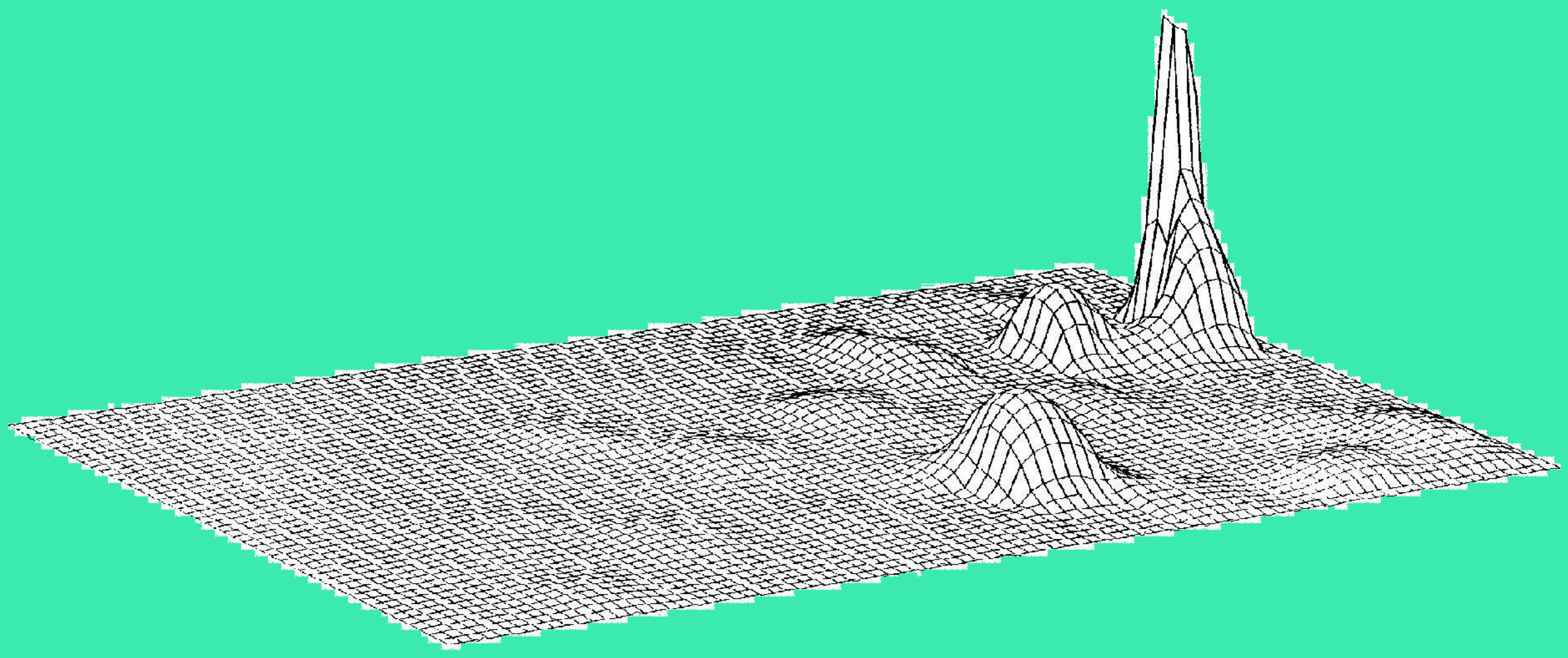


A piecewise continuous surface

Population Density in Kansas

by County

Each county still contains the same number of people



A smooth continuous surface, with population pycnophylactically redistributed

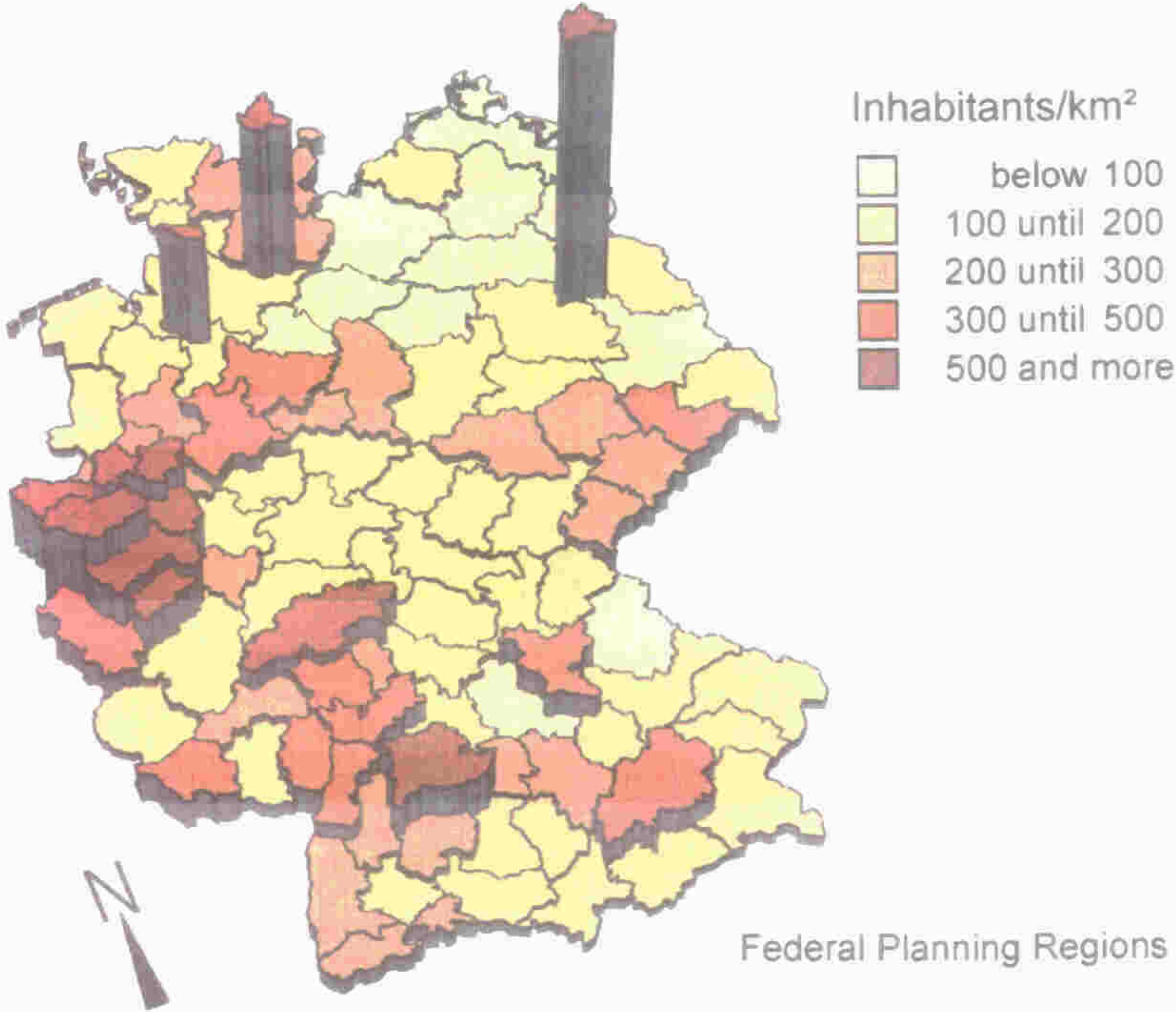
Another Example

This time using population data by Federal Planning Regions for Germany.

First the data are represented in a perspective view of a bivariate histogram.

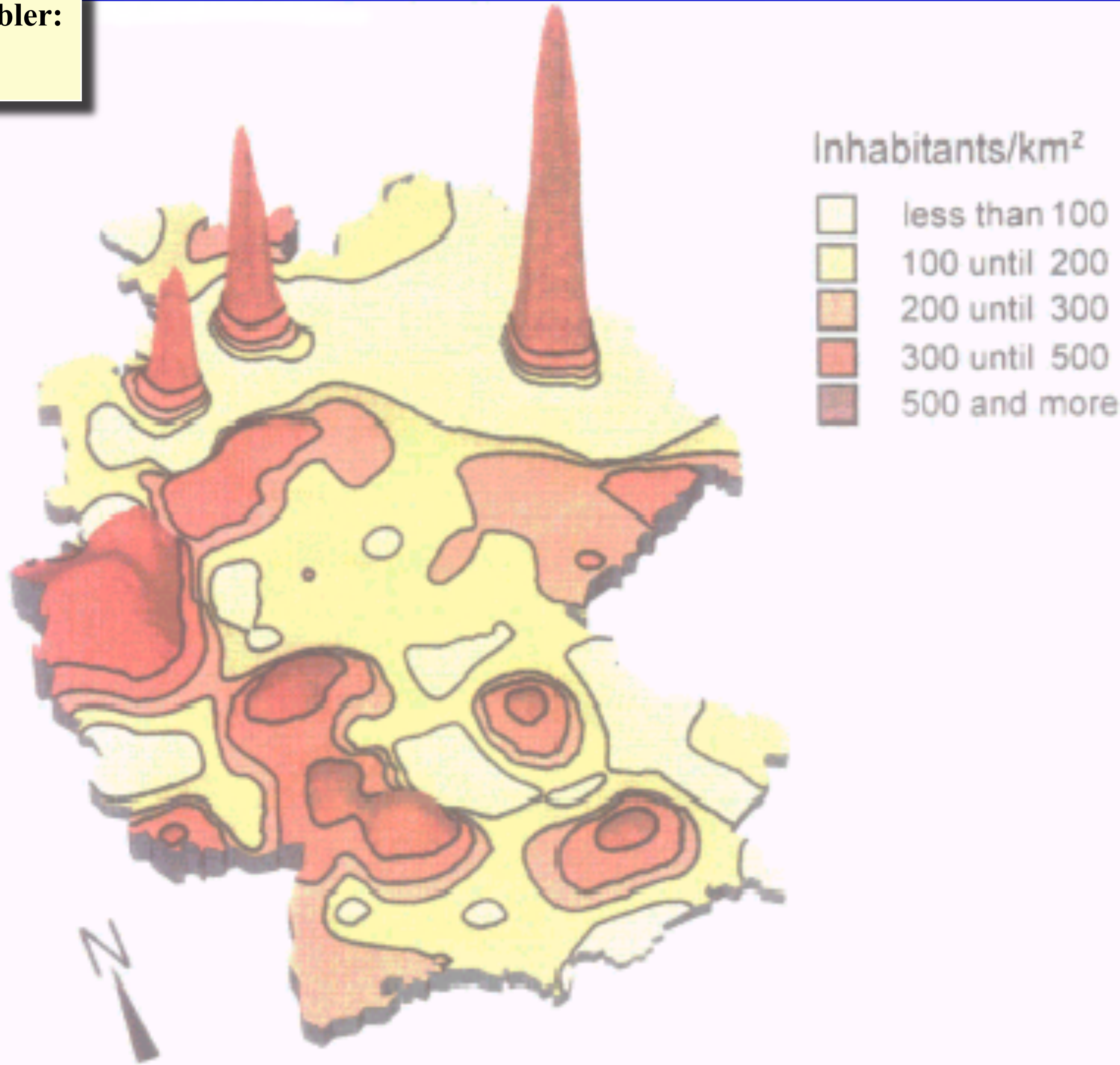
This is followed by a similar view of the continuous population density distribution.

Courtesy of Wolf Rase in Bonn.



Inhabitants/km²
below 100
100 until 200
200 until 300
300 until 500
500 and more
N
Federal Planning Regions

Waldo Tobler:



How Pycnophylactic Reallocation Works

Philosophically it is based on the notion that people are gregarious, influence each other, are mobile, and tend to congregate.

This leads to neighboring and adjacent places being similar.

Mathematically this translates into a smoothness criterion (with small partial derivatives).

It applies to any data exhibiting positive spatial autocorrelation.

Mass Preserving Reallocation Using Areal Data

First define the primary condition for mass preservation. This is the required invertibility condition for any method of areal information redistribution:

$$\iint_{R_i} f(x,y)\ dx\ dy = V_i \quad \text{for all i,}$$

where V_i denotes the value (population in the present context) in region R_i (a subnational polygon).

Next constrain the resulting surface to be smooth by requiring neighboring places to have similar values. This is an assumption about spatial demographic processes, a form of geographic insight capturing the notion that most people are gregarious. Laplacian smoothness, the simplest kind, is obtained by minimizing:

$$\iint_{R} \left(\frac{\partial f}{\partial x}^2 + \frac{\partial f}{\partial y}^2 \right) dx\ dy$$

where R is the set of all regions. The boundary condition is

$$\frac{\partial f}{\partial \eta} = 0 .$$

W. Tobler, 1979, "Smooth Pycnophylactic Interpolation for Geographical Regions", *J. Am. Stat. Assn.*, 74(367):519-536.

The minimum of the integral

The solution of the integral smoothness equation is given by the LaPlace equation

$$\nabla^2 X + \nabla^2 Y = 0.$$

This says that the neighboring locations have similar values - or in a raster, that the central value is the average of those surrounding it. This immediately yields a computational algorithm.

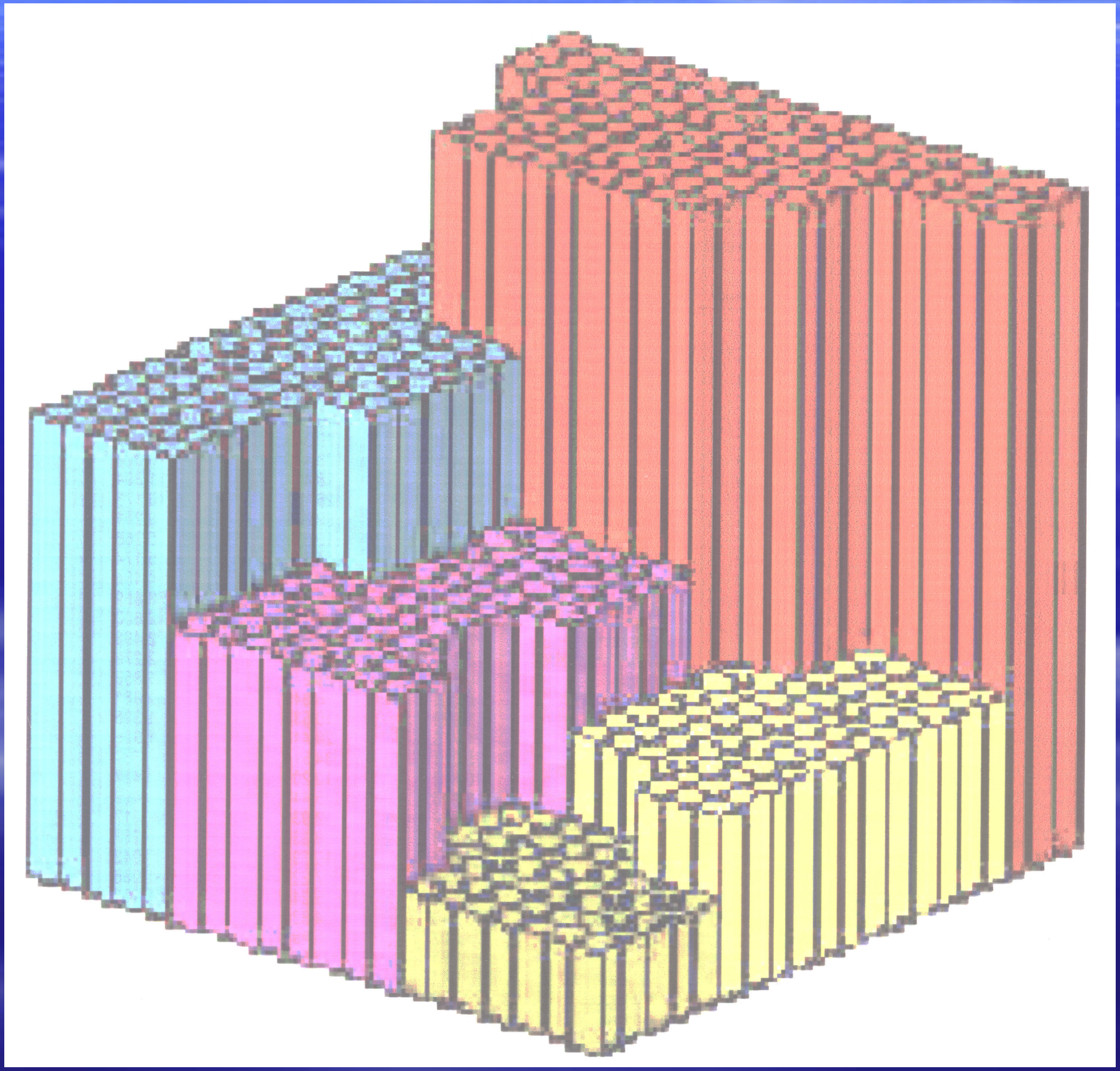
What the Mathematics Means

Imagine that each unit is built up of colored clay, with a different color for each unit.

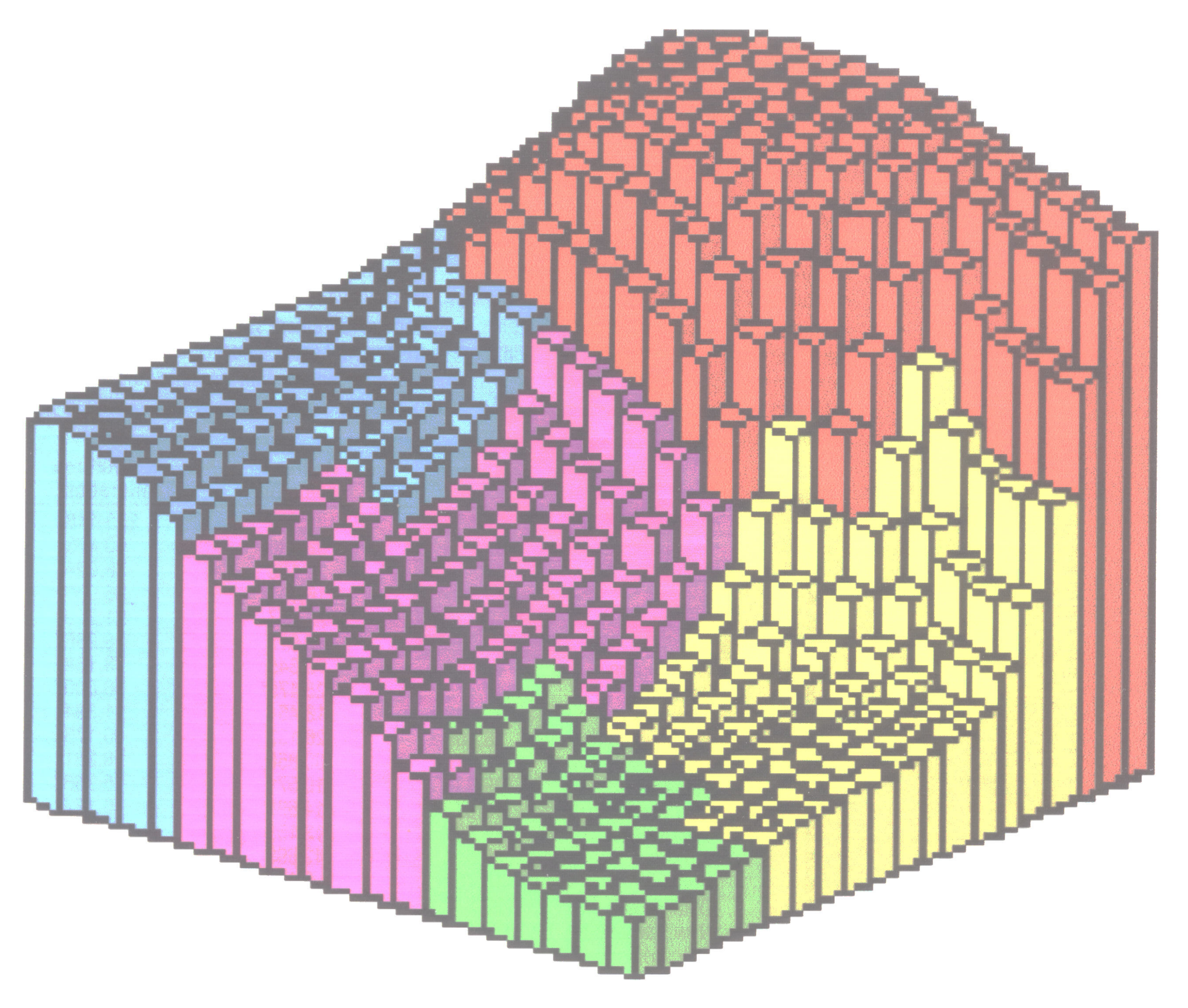
The volume of clay represents the number of people, say, and the height represents the density.

In order to obtain smooth densities a spatula is used, but no clay is allowed to move from one unit into another. Color mixing is not allowed.

Colored Clay Before Smoothing



Colored Clay After Smoothing



The Smoothing is done using an iterative process.

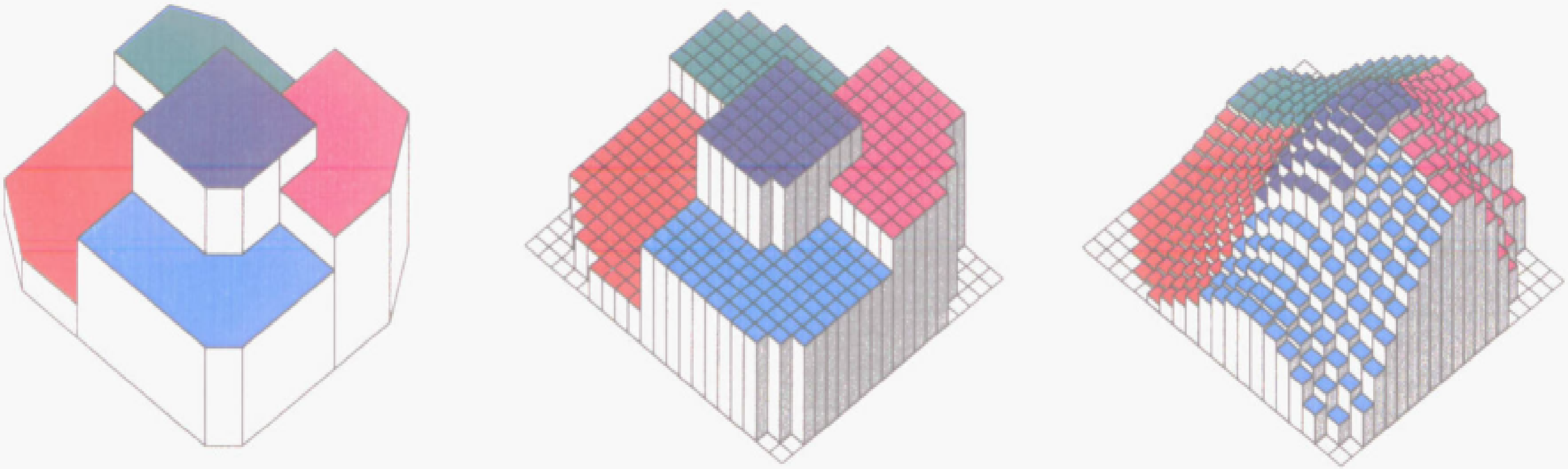
The first step is to “rasterize” the region. Then the smoothing is done on this raster, all the while maintaining the “population”.

The number of iteration steps depends on the size of the largest region, in raster units.

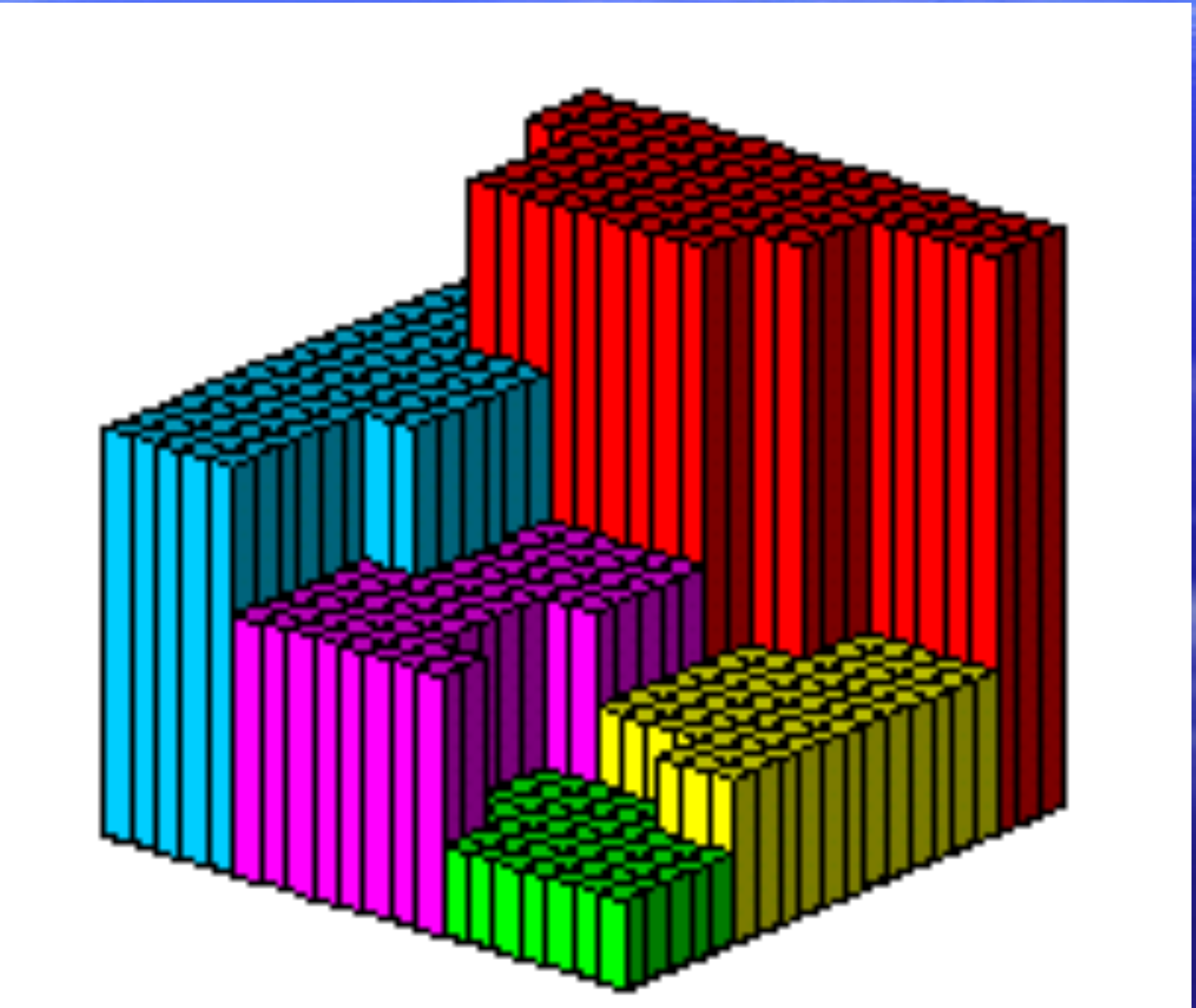
That is because the smoothing must cross from edge to edge of the largest region. The finer the raster, the higher the resolution and the longer the iteration time.

Left to right

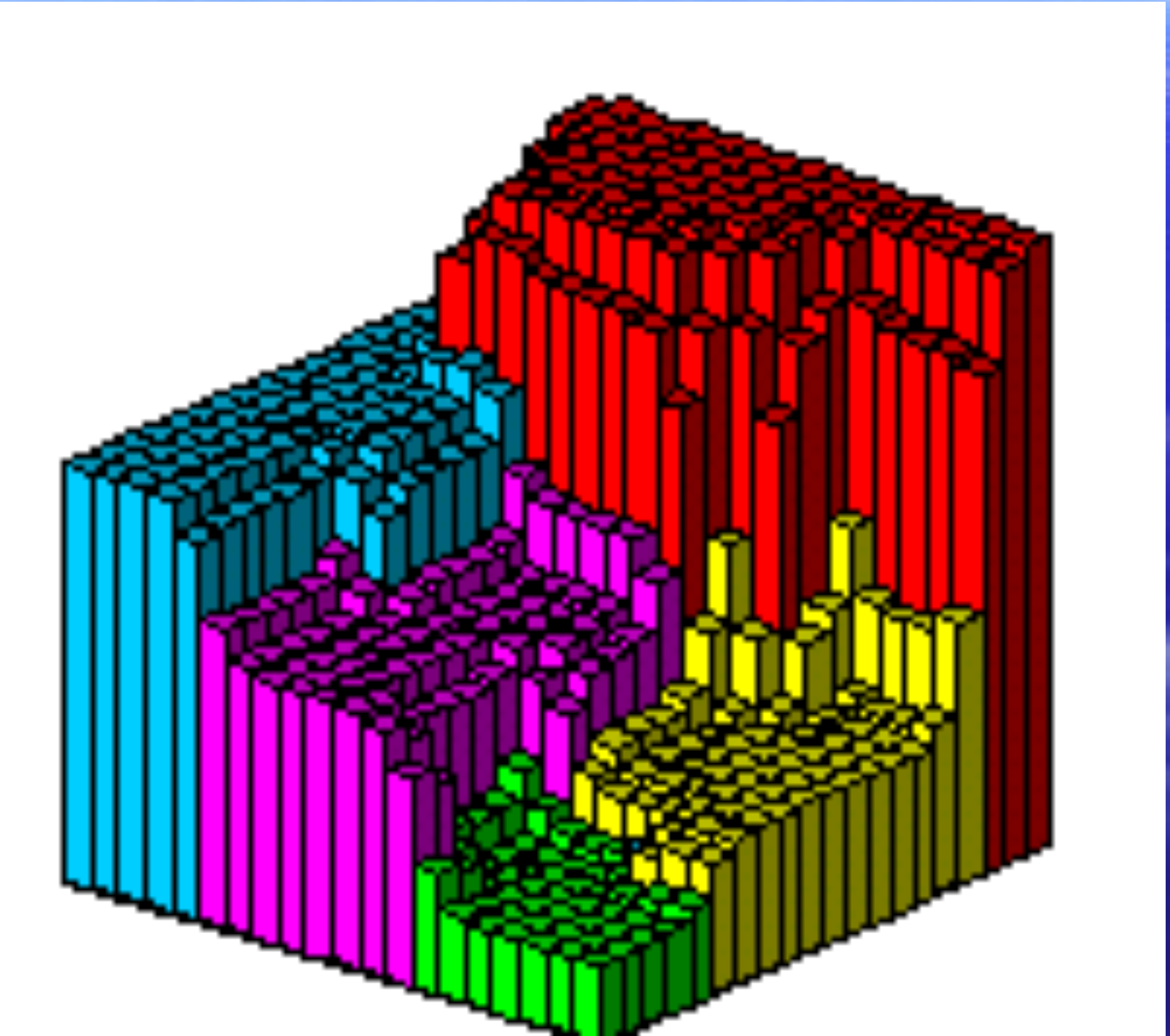
1. Data Polygons
2. Rasterized
3. Smoothed



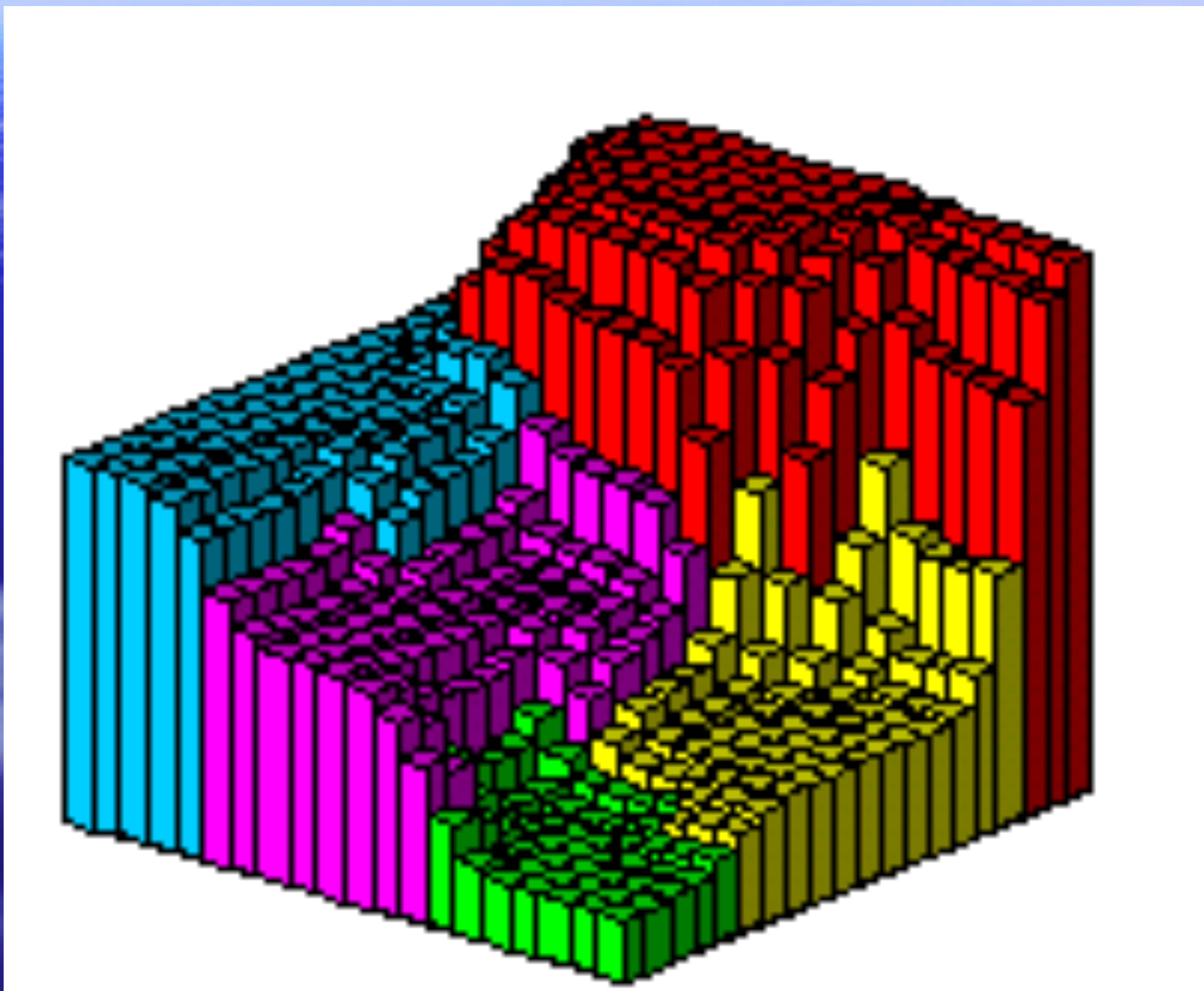
Zero Iterations



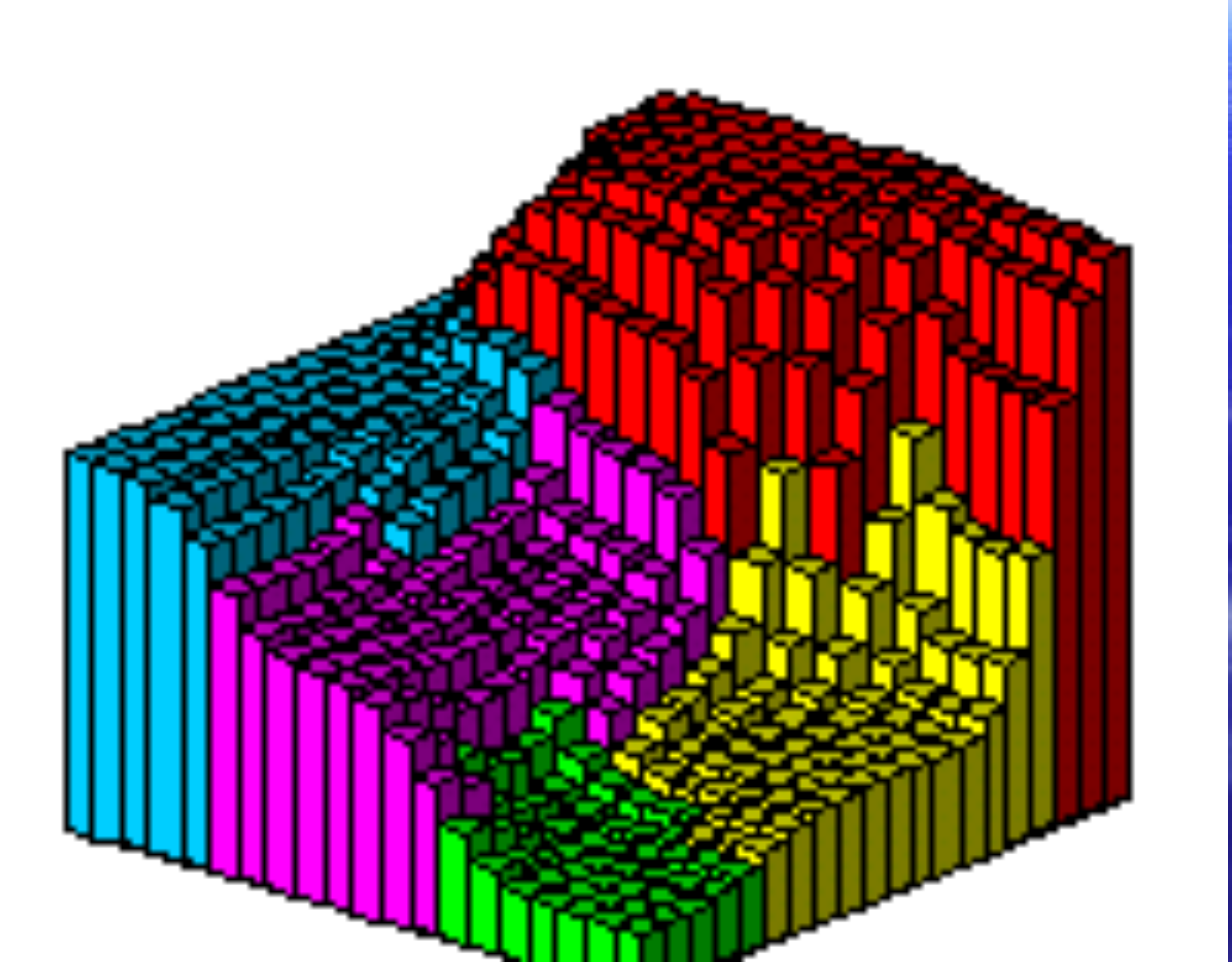
Five Iterations



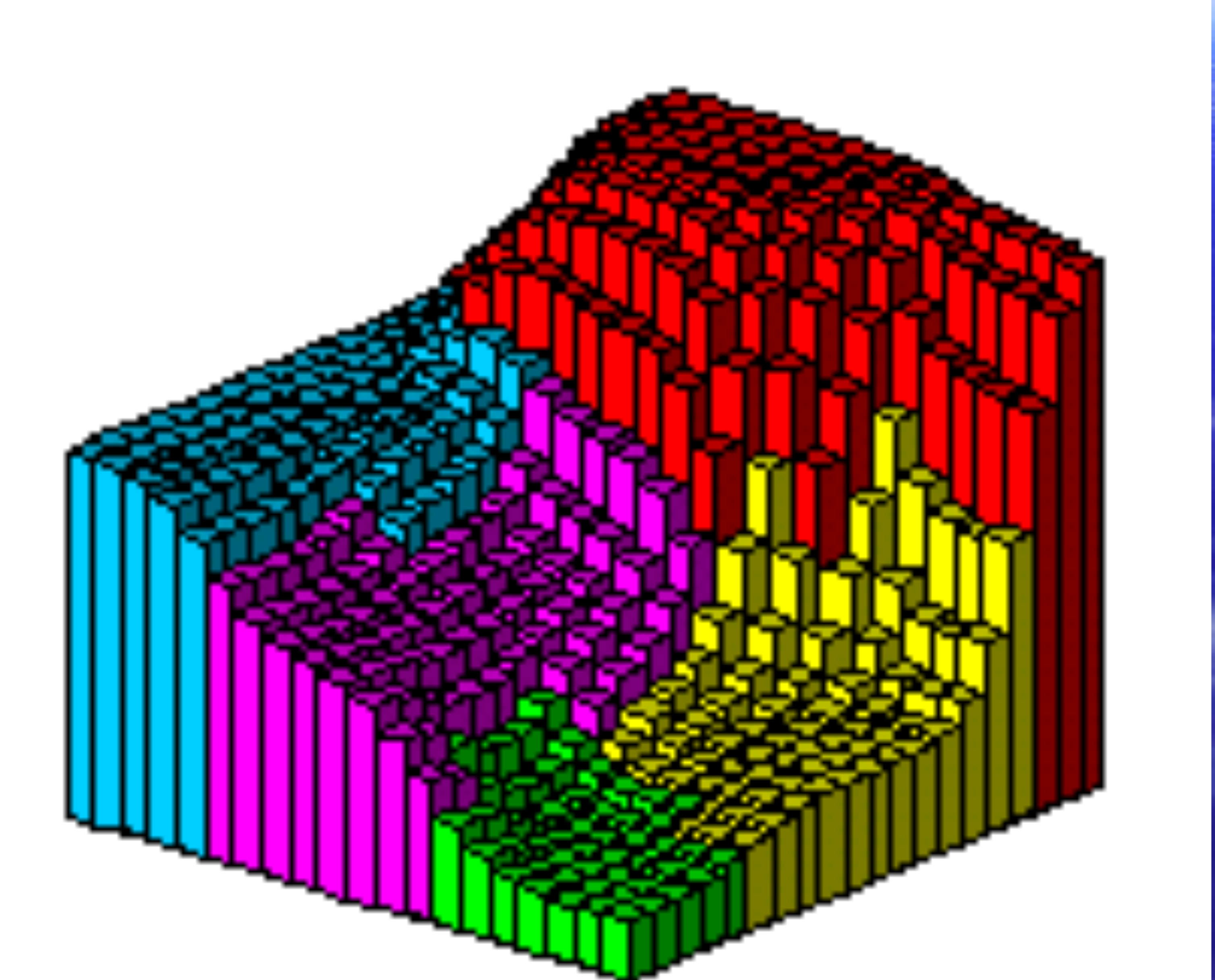
Ten Iterations



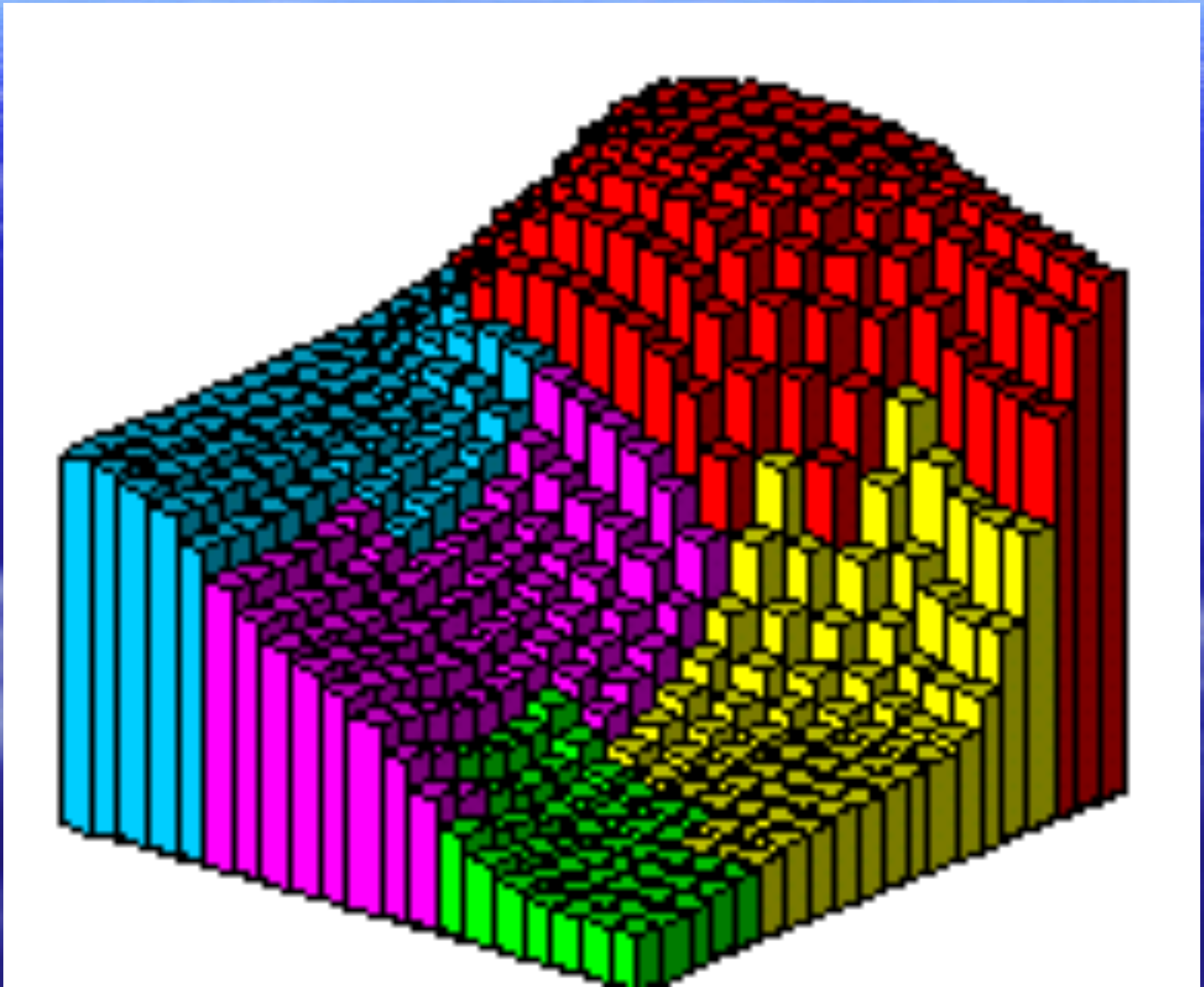
Fifteen Iterations

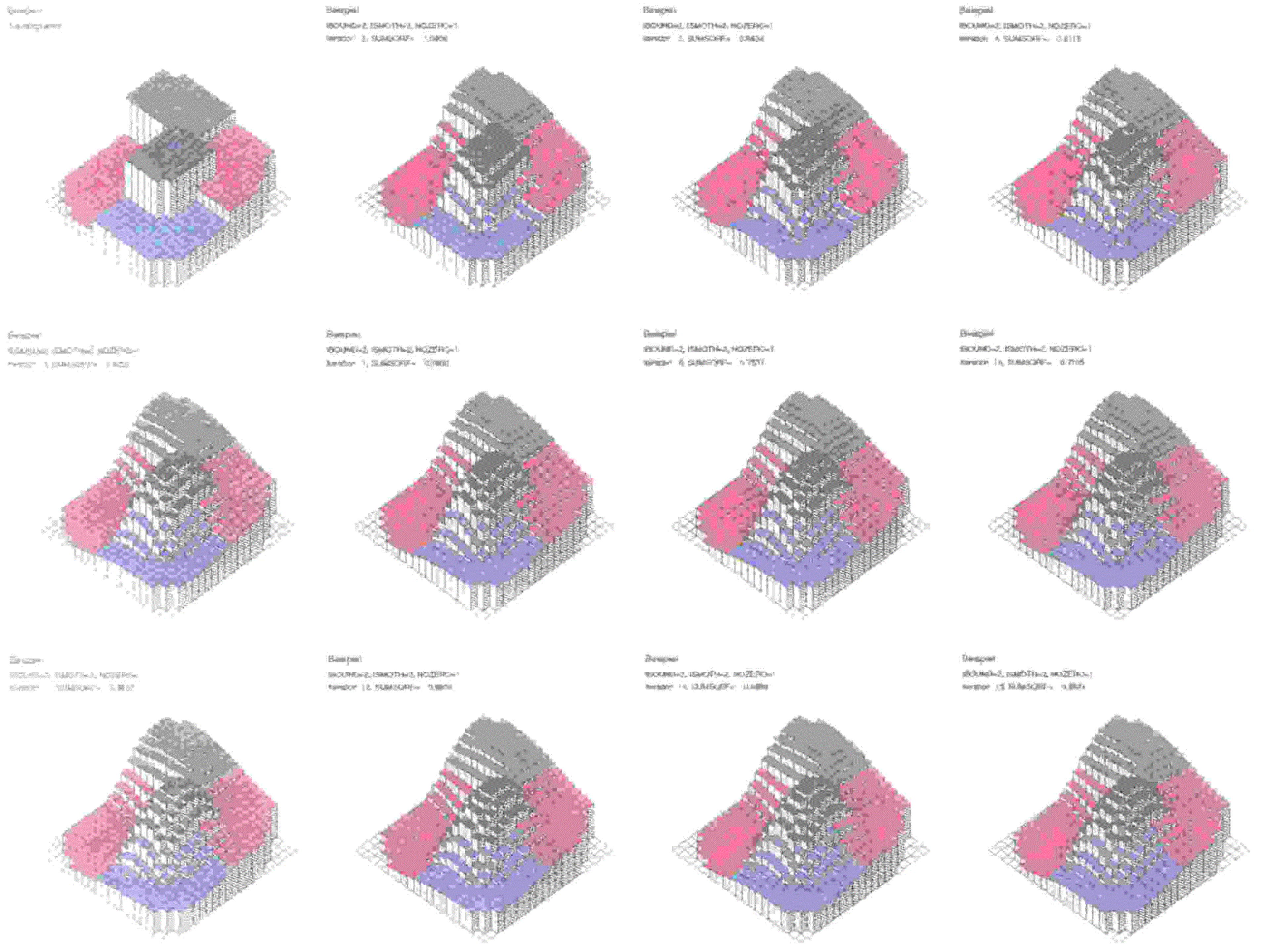


Twenty Iterations



Twenty Five Iterations



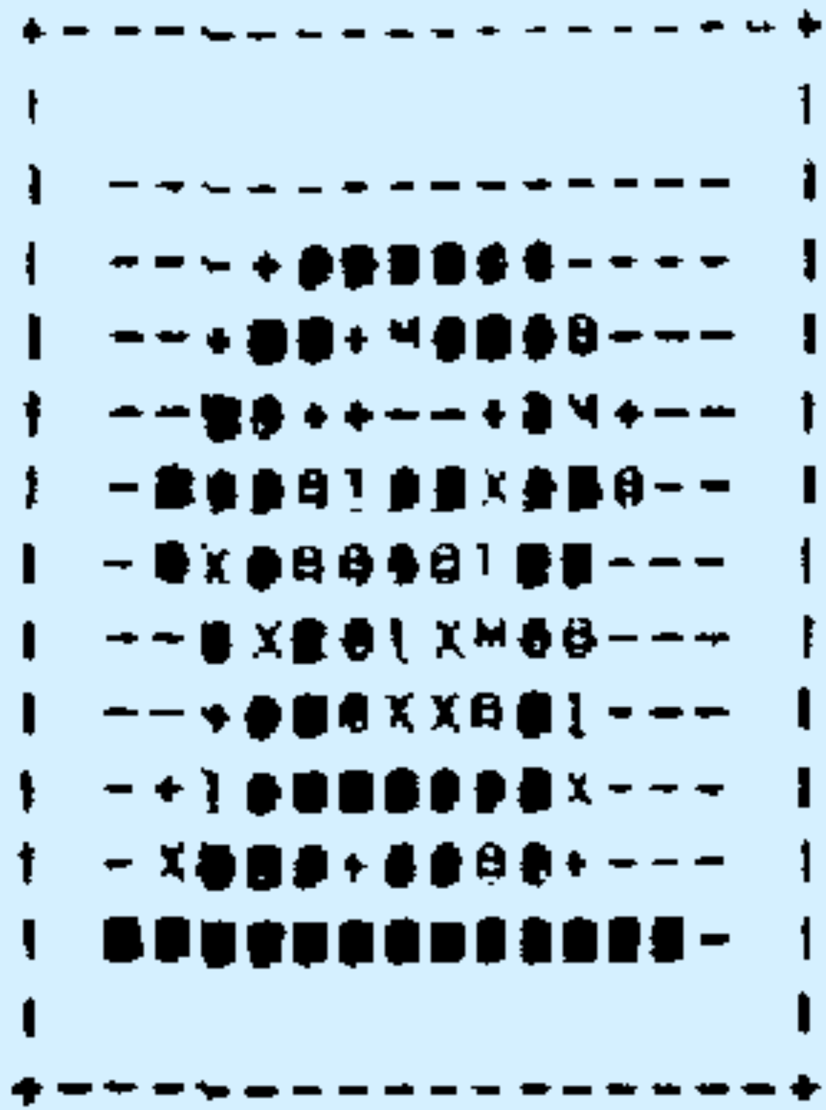

Beispiel
IBOUND=2, ISMOTH=2, NOZERO=1

Pycnophylactic reallocation also works for data assembled within individual cells of a lattice or grid although this was not the design objective.

For example, data given within pixels.

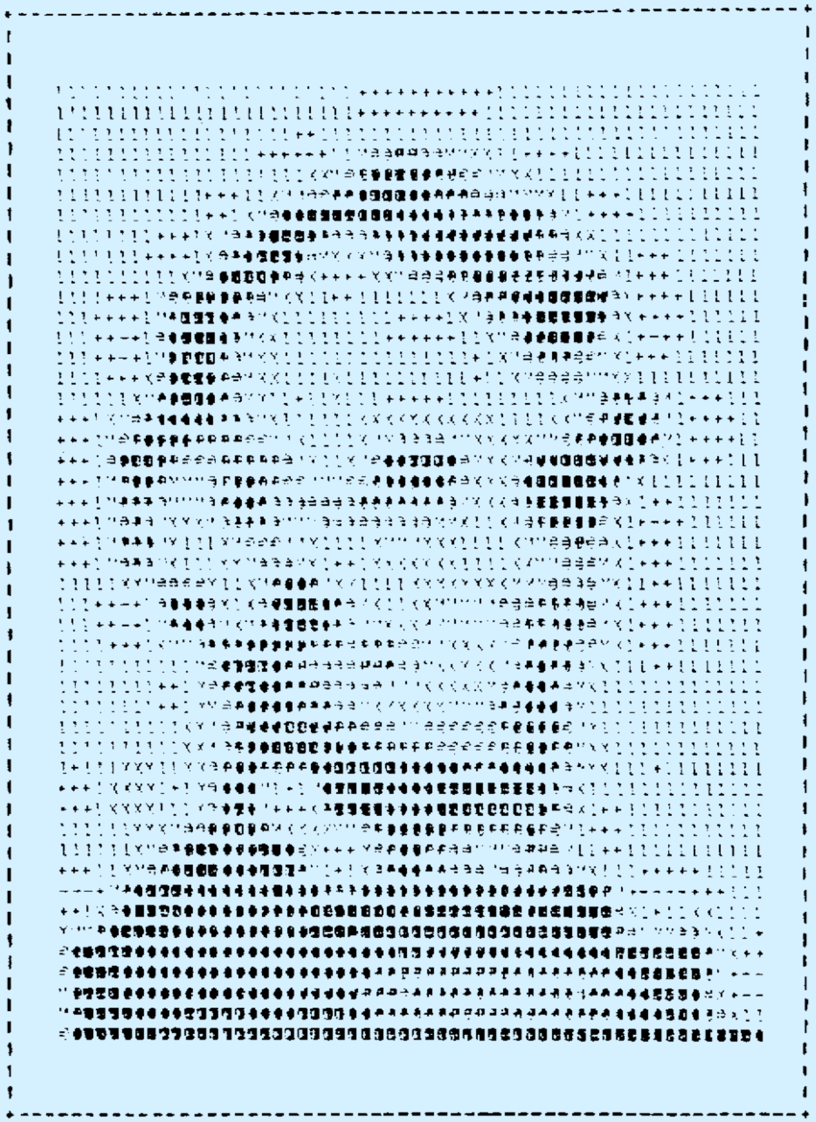
Not between pixels which results in a different effect. But values in neighboring pixels are taken into account <u>within</u> a pixel by the smoothness criteria.

An Image Processing Example
A 20 by 14 Image



Quadrupled to 80 by 56

but with the same total "mass"



Smoothing & Boundary Conditions

The procedure can use different smoothing criteria.

There is a choice between LaPlacian and biharmonic smoothing.

As the solution to a partial differential equation it is also necessary to specify boundary conditions.

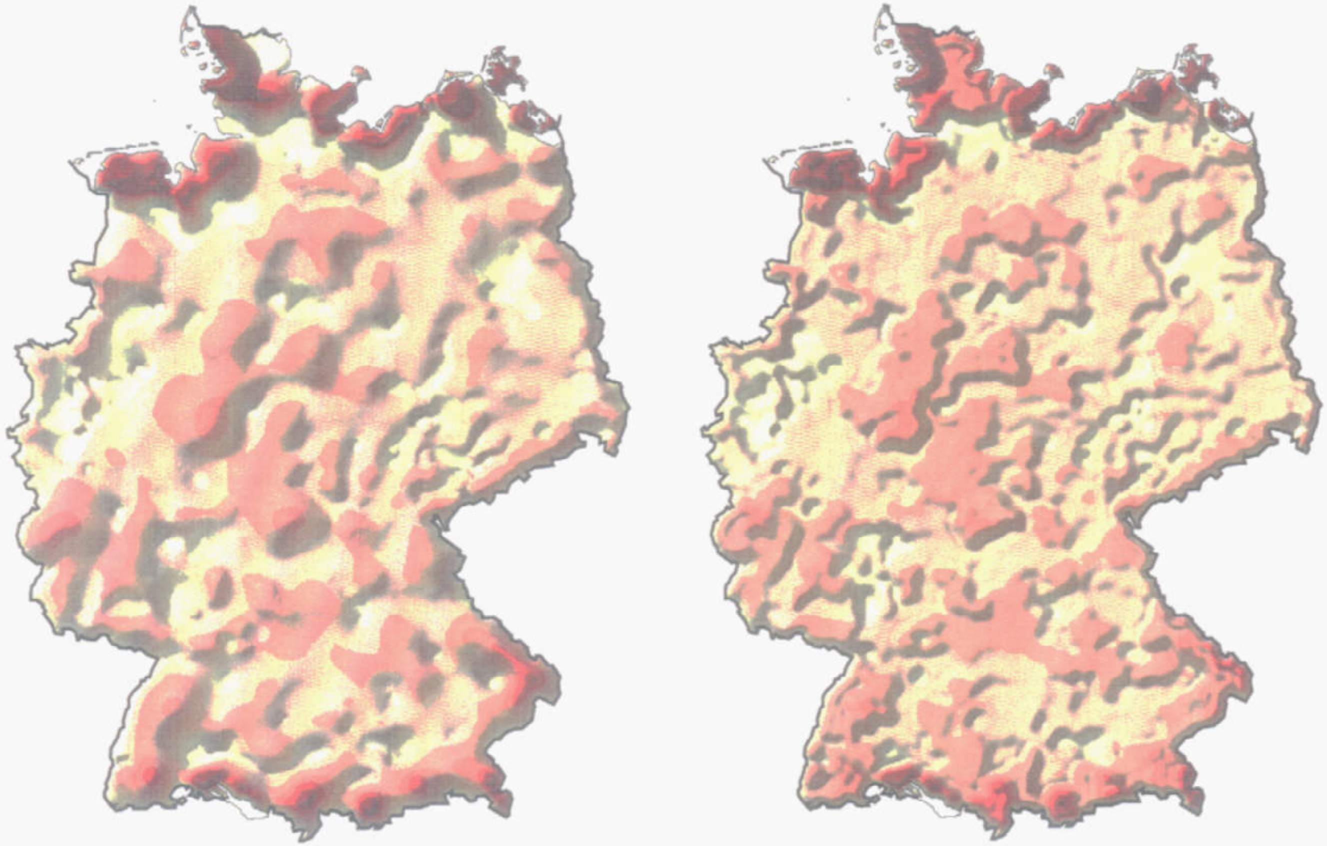
The Dirichlet condition specifies the value at the boundary. The Neumann condition specifies the gradient at the boundary.

LaPlacian & Biharmonic Smoothing

Dirichlet Boundary Condition

Landschaftliche Attraktivität

Stetigkeit:	Laplace	Stetigkeit:	Biharmonisch
Grenzbedingung:	Dirichlet	Grenzbedingung:	Dirichlet
Iterationen:	100	Iterationen:	50
Restfehler:	0.3142 %	Restfehler:	0.0099 %



a) Stetigkeit nach Laplace-Gleichung

b) Stetigkeit nach biharmonischer Gleichung

Abb. 2-29 Pyknophylaktische Interpolation mit Dirichlet-Grenzbedingung

LaPlacian & Biharmonic Smoothing

Neumann Boundary Condition

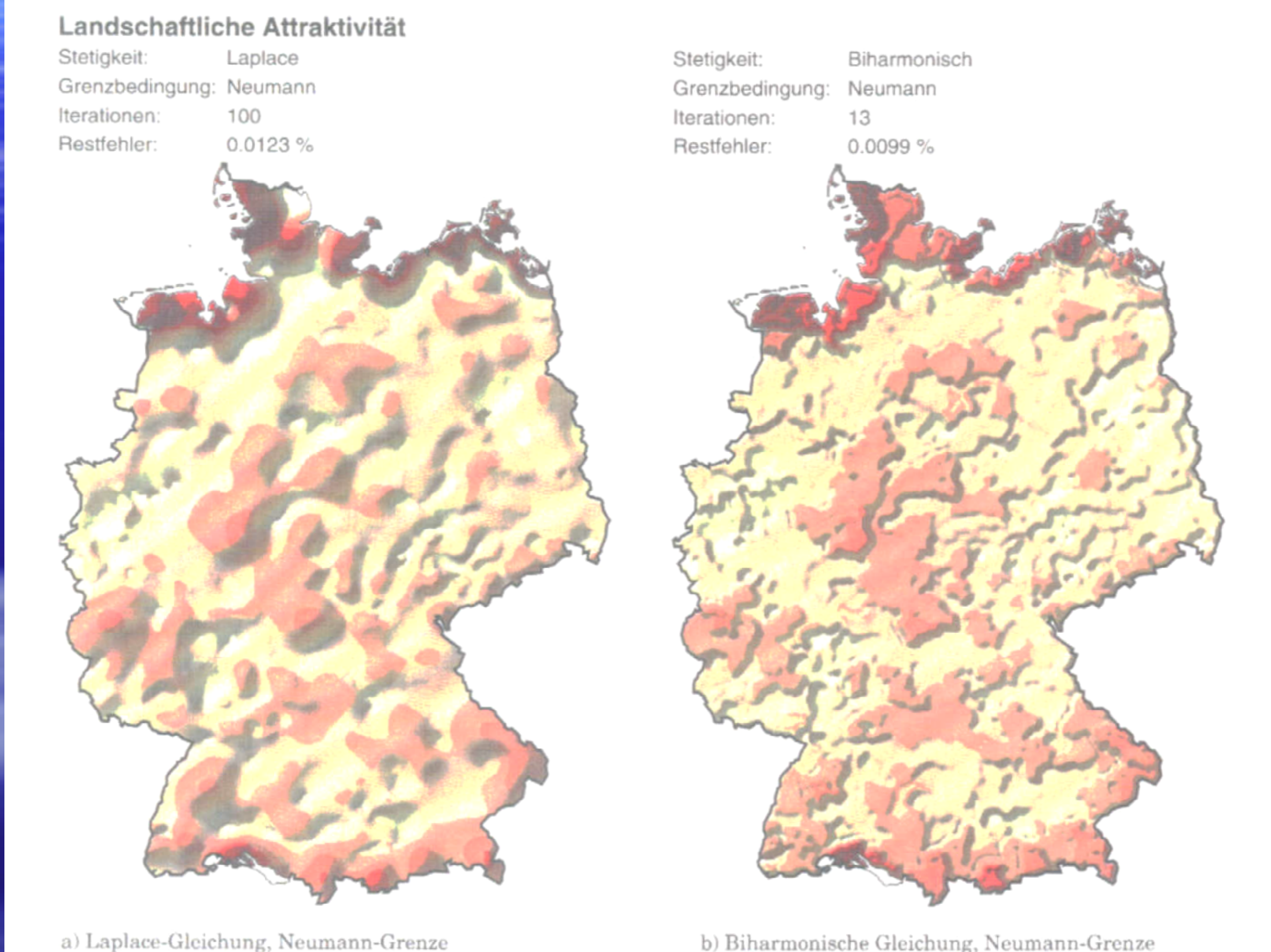
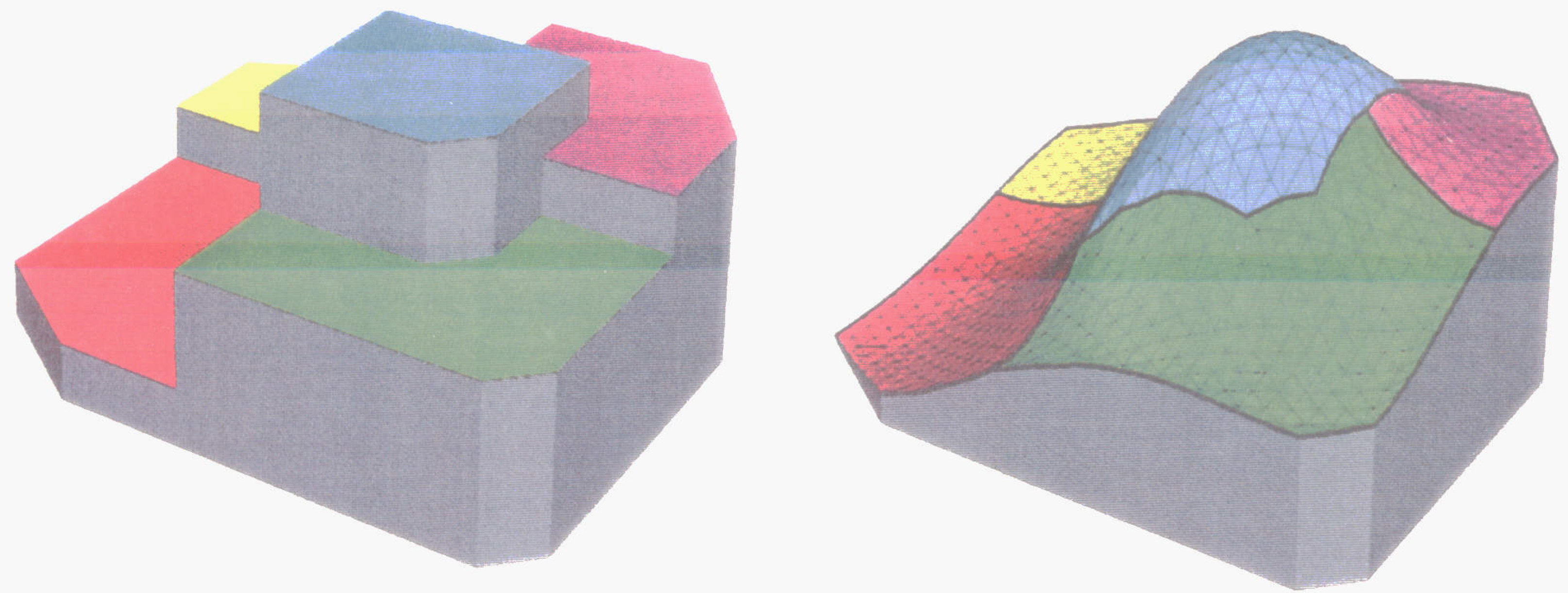


Abb. 2-30 Pyknophylaktische Interpolation mit Neumann-Grenzbedingung

Finite Elements Also Work

The original and smoothed using triangles.

Wolf-Dieter Rase, 2001, "Volume-preserving interpolation of a smooth surface from polygon-related data", J. Geograph. Syst., 3: 199-213



a) Perspective view of choropleth map

b) Regular grid of triangles

When you have more data

When data from other sources (e.g., aerial photographs) is available it may be appropriate to redefine the polygons to exclude certain areas. For example when the phenomenon of interest consists of population then lakes, forests, and industrial zones normally would be devoid of population. The new set of polygons would exclude these - remember to use the Dirichlet condition at the boundary of these areas with a zero value. Then proceed with these new polygons as in the usual case. This can be refined to include additional polygons with diminished or enhanced population values. This is known in cartography as the dasymetric method. The problem and solution are not changed, but the polygons are.

Another Important Advantage of Mass Preserving Reallocation

A frequent problem is the reassignment of observations from one set of collection units to a different set, when the two sets are not nested nor compatible. For example converting the number of children observed by census tract to a count by school district. Boundaries also change over time, requiring reallocation for compatibility.

The density values obtained using the smooth pycnophylactic method allow an estimate to be made rather simply. A "cookie cutter" can cut the continuous clay surface into the new zones with subsequent addition (summation) to get the count.

Does it make a difference?

There have been few comparisons of mass preserving areal reallocation and point based interpolation.

The following table compares the mass preservation property of several point based interpolators, based on the German data.

This was done by Wolf Rase in his dissertation.

Comparing Volume Preservation

using different interpolations (W. Rase)

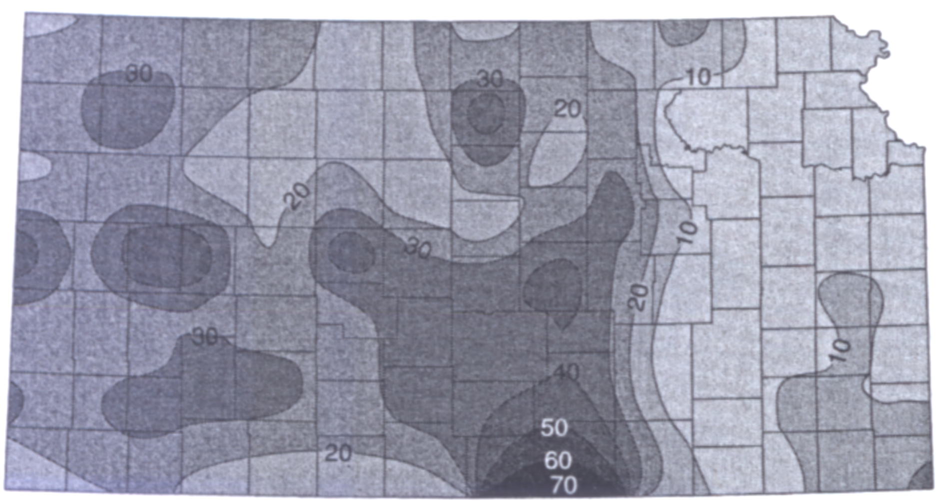
Interpolations-verfahren	Absolute Abweichung vom Variablenwert in Prozent			Relative Varianz
Gitter, ohne Partionierung	Minimum	Mittelwert	Maximum	Prozent
Shepard	0.01	10.65	1013.39	3.51
Hardy (Q=0.5, R=0.21)	0.00	11.07	1515.45	4.23
Gitter, konvexe Hülle				
Renka, linear	0.00	17.39	3192.25	10.66
Renka, globale Gradientenschatzung	0.01	13.26	1113.36	6.38
Renka, lokale Gradientenschatzung	0.04	14.65	2463.10	7.79
Renka mit QUADSF (Spath)	0.00	12.77	998.08	5.95
Montefusco/Casciola	0.01	11.52	1114.73	4.82
Akima	0.00	11.59	1389.10	5.08
Gitter, konkave Hülle				
Renka, globale Gradientenschatzung	0.01	10.45	1112.81	3.60
Renka, lokale Gradientenschatzung	0.01	12.32	2658.03	6.49
Renka mit QUADSF (Spath)	0.00	10.93	997.60	4.81
Fortgesetzte Teilung des TIN				
Shepard	0.03	6.64	2147.73	0.52
Renka, globale Gradientenschatzung	0.03	2.75	2644.66	0.16
Renka, lokale Gradientenschatzung	0.01	0.58	4111.18	0.16
Renka mit QUADSF (Spath)	0.01	4.13	2632.20	0.24
Pyknophylaktische Interpolation	0.10	1.89	364.18	0.02

Two types of isopleth interpolation compared.

Left pycnophylactic reallocation, right punctual Kriging from centroids.

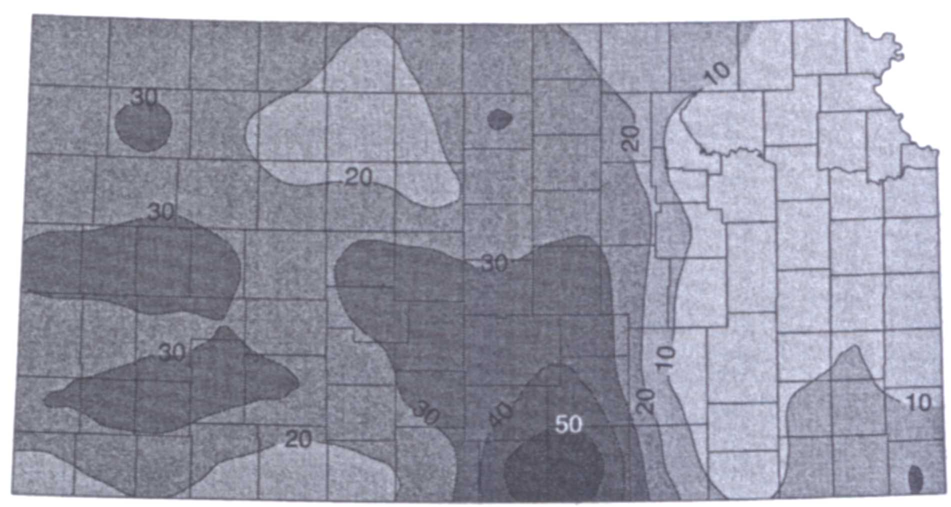
Figure 8.13 (p. 150) of T. Slocum, "Thematic Cartography and Visualization", Prentice Hall, 1999.

Wheat Harvested in Kansas, 1993
(pycnophylactic)



Contour Lines Represent Percent of Land Area

Wheat Harvested in Kansas, 1993
(kriging)



Contour Lines Represent Percent of Land Area

More Tests Needed

Tests are also needed to determine the viability of pycnophlyactic reallocation when converting from one set of areas to another.

To date no adequate tests of this procedure have been reported.

In essence these would test the adequacy of the hypothesis implying spatial autocorrelation - that neighboring places have similar values.

I appreciate your attention and thank you.