SPATIAL INTERACTION PATTERNS *

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ABSTRACT

An algebraic examination of spatial models leads to the conclusion that a convenient description of the pattern of flows implicit in a geographical interaction table is obtained by displaying a field of vectors computed from the relative net exchanges. The vector field approximates the gradient of a scalar potential, and this may be invoked to explain the flows. The method can be applied to asymmetrical tables of a non-geographical nature.

Introduction

Empirical measurements of the interaction between geographical areas are often and conveniently represented by "from-to" tables, usually with asymmetric entries. Many mathematical models offered as descriptors of these geographical interaction patterns do not treat this situation adequately. Specifically, they quite frequently can predict only symmetrical interaction tables, a glaring contrast to the empirical observations. In the present essay an attempt is made to overcome this difficulty through the introduction of a flow field, which one may wish to think of as a "wind." This wind is interpreted as facilitating interaction in particular directions. The algebra allows one to estimate the components of this hypothetical flow field from the empirical interaction tables. Plotting the flow field provides a simple, convenient, and dramatic cartographic representation of the asymmetry of the exchanges, even for extremely large tables of interaction data. A table of county-to-county interactions in the United States, for example, would yield nearly 10^7 numbers, an incomprehensible amount. A flow field, on the other hand, showing these data as a set of vectors might be more tractable. Going one step further, it should be possible to infer an estimate of the forcing function, the "pressure," which might be said to have given rise to the interaction asymmetry. Data collected for several time periods may allow one to infer the dynamics of the relation of the forcing function and the flows.

Background

The foregoing objectives were motivated by previous papers in which geographical locations were predicted from empirical interaction data by inverting models which contained spatial separations as one of the explanatory variables [1,2]. The resulting spatial separations were then converted to latitude and longitude positions by a procedure analogous to trilateration, as practiced in geodesy. The empirical data in each instance were indicators of the amount of interaction between the locations in question. These interactions were given in the form of matrices, M_{ij}, in which the rows are the "from" places and the columns the destination places. For example, if M_{ij} is the amount of migration from place i to place j, then the social gravity model predicts that $M_{ij} = kP_iP_jf(d_{ij})$, where the P's denote the populations, d stands for distance, and k is a scale factor. Then the inversion is

$$d_{ij} = f^{-1}\left(\frac{M_{ij}}{kP_iP_j}\right).$$

From the adjustment procedures used in surveying one learns how to calculate the latitude and longitude coordinates of positions when their separations have been measured [3]. A similar procedure has recently been used in psychology [4]. The social gravity model of course is symmetrical in the sense that if $d_{ij} = d_{ji}$ then M_{ij} must equal M_{ji}, and the converse. In practice, however, interaction matrices are asymmetrical, and $M_{ij} \neq M_{ji}$. This would imply, if the model is inverted as was done above, that $d_{ij} \neq d_{ji}$, with the consequence that the trilateration solutions can result in more than one geometrical configuration or that the standard errors of the position determination are increased [5]. In order to overcome this difficulty it is natural to introduce a wind, or current of some type, which facilitates interaction in particular directions. This vector field is to be estimated from the empirical data, and of course reflects their influence. At the moment the wind need not be given any interpretation other than that of a mathematical artifact which allows the problem to be solved. Later we can look for independent evidence which might confirm (or deny) its existence.

Algebraic Development

As a simple example, postulate that a travel effort (time, cost, etc.) t_{ij} is aided by a flow $\mathbf{c}_{ij}$ in the direction of movement from place i to place j. Then we can write that $t_{ij} = d_{ij} / (r + \mathbf{c}_{ij})$ where r is a rate of travel, independent of position and of direction, and is in the same units as **c**. An interpretation might be that t_{ij} is travel time for someone rowing on a lake, r is the rowing speed in meters per second, and **c** is a current in the water; or that j is uphill from i, and that this results in a difference in travel speed; or that there exists a grain, as when skins are placed underneath skis, and movement is rendered easier in one direction. Whatever the interpretation, solving for **c** one obtains

$$\vec{c}_{ij} = r\,\frac{t_{ji} - t_{ij}}{t_{ij} + t_{ji}}.$$

Here use has been made of the relation $\mathbf{c}_{ij} = -\,\mathbf{c}_{ji}$ which must hold for currents, and of $t_{ji} = d_{ji} / (r + \mathbf{c}_{ji})$.

The same argument can be applied to the gravity model. Substitute t for d in that model, with $f(t) = t^{-1}_{ij}$ for simplicity,

$$M_{ij} = k\frac{P_iP_j}{t_{ij}} = k\frac{P_iP_j}{d_{ij}}(r + \vec{c}_{ij}),$$

and

$$M_{ji} = k\frac{P_jP_i}{t_{ji}} = k\frac{P_jP_i}{d_{ji}}(r + \vec{c}_{ji}) = k\frac{P_iP_j}{d_{ij}}(r - \vec{c}_{ij}).$$

Solving for $\mathbf{c}_{ij}$, recalling that $d_{ij} = d_{ji}$, one finds

$$\vec{c}_{ij} = r\frac{M_{ij} - M_{ji}}{M_{ij} + M_{ji}}.$$

A convenient choice of units will make $r \equiv 1$ and then

$$\vec{c}_{ij} = \frac{M_{ij} - M_{ji}}{M_{ij} + M_{ji}}.$$

It is encouraging that this quantity has already been found useful in studies of migration, albeit without the present derivation [6, 7]. The original objective, inversion of the model, follows immediately:

$$d_{ij} = \frac{P_iP_j}{M_{ij}}\left(r + r\frac{M_{ij} - M_{ji}}{M_{ij} + M_{ji}}\right).$$

Reversing i and j does not change this quantity. Algebraically

$$d_{ij} = \frac{2rP_iP_j}{M_{ij} + M_{ji}}.$$

which is the same result as would have been obtained if

$$\frac{M_{ij} + M_{ji}}{2} = k \frac{P_i P_j}{d_{ij}}$$

had been assumed initially. A solution to the original problem has thus been achieved, in the sense that any asymmetric interaction table can be made to yield a unique distance estimate to be used in further computations involving locations.

Two difficulties remain. First, only one interaction model has been examined. Secondly, can a reasonable interpretation be provided for the $\mathbf{c}_{ij}$ when the interaction consists of, say, telephone calls between exchanges?

In the first instance, a more general gravity model might be written using

$$\frac{M_{ij} + M_{ji}}{2} = k \frac{P_i P_j}{d_{ij}^{\alpha}} ,$$

the development of which is straightforward. In a similar vein, an exponential model

$$M_{ij} = k(P_i + P_j)\ e^{-b\,d_{ij}/(r+\vec{c}_{ij})}$$

yields

$$\vec{c}_{ij} = r \left(\frac{\ln M_{ji} - \ln M_{ij}}{\ln M_{ij} + \ln M_{ji} - 2 \ln k\ (P_i + P_j)} \right)$$

and this is a much more complicated result. One could continue further by, for example, considering the entropy model elaborated by, Wilson, or the migration model published by Lowry [8, 9] . These models are in fact already more general in that they do yield asymmetrical interaction tables, but they also require supplementary information before one can solve for the distances. The Lowry model is

$$M_{ij} = k \frac{U_i}{U_j} \frac{W_j}{W_i} \frac{P_i P_j}{d_{ij}^{\alpha}} ,$$

where U is related to unemployment and W to wages. This can be rewritten as

$$M_{ij} = k \frac{a_i}{a_j} \frac{P_i P_j}{d_{ij}^{\alpha}}$$

and implies, if $d_{ij} = d_{ji}$ that (using $a_i = U_i/W_i$, $a_j = U_j/W_j$)

$$\frac{M_{ij}}{M_{ji}} = \left(\frac{a_i}{a_j} \right)^2$$

and also (solve for a_j and substitute) that

$$M_{ji} = (kPiP_j d^{-\alpha}_{ij})^2 M^{-1}_{ij} \quad \text{or} \quad (U_j/U_i)(W_i/W_j) = k\, P_i\, P_j\, f(d_{ij})$$

Thus the second half of the interaction table carries no information. Furthermore, if the distances are known, one can infer the wage and unemployment ratios from the empirical migration data. Such a result has recently also been achieved for another model by Cordey-Hayes [10-12]. In the present instance there are n(n - 1)/2 equations of the form

$$k^{-1}\, P^{-1}_i\, P^{-1}_j\, d^{-\alpha}_{ij}\, M_{ij} = a_i/a_j,$$

and n unknowns, the a_i's. The system is overdetermined unless some of the equations can be shown to be dependent.

In a comparable manner, given only an empirical interaction table, then the row sums $O_i = \Sigma\, M_{ij}$ and column sums $D_j = \Sigma\, M_{ij}$ can all be computed. The simplest sort of model is then that $M_{ij} = k\, O_i\, D_j\, f(d_{ij})$, and $M_{ji} = k\, O_j\, D_i\, f(d_{ji})$ where the origin and destination sums now take the place of the populations. In order to obtain a consistent value for $d_{ij} = d_{ji}$ it is necessary that $M_{ij}/M_{ji} = O_iD_j/O_jD_i$, and this is a hypothesis that can be tested.

Another interesting model has been proposed by Somermeijer [13]. This is

$$M_{ij} = (a - bQ_{ij}) \frac{P_i P_j}{d_{ij}^{\alpha}} .$$

Here Q_{ij} is the difference in attractiveness between areas i and j, $Q_{ij} = A_j - A_i$. Clearly $Q_{ij} = - Q_{ji}$, solving for this quantity using

$$M_{ji} = (a - bQ_{ji}) \frac{P_j P_i}{d_{ji}^{\alpha}} ,$$

and adding, then subtracting, the equations for M_{ij} and M_{ji}, one finds

$$Q_{ij} = \frac{a}{b} \left(\frac{M_{ji} - M_{ij}}{M_{ij} + M_{ji}} \right) .$$

This is a very interesting relation because, although there is much speculation in the literature, no one really knows by how much areas differ in attractivity. The model allows an estimate to be made of this quantity. One notices that $Q_{ii} = 0$, although usually $M_{ii} \neq 0$, and a desirable property would be that $Q_{ij} = Q_{iik} + Q_{kj}$ for all i, j, k. In this case the attractivity of area j, call it A_j would simply be $A_j = A_i + Q_{ij}$ for some base level A_i. If this relation does not hold for all i and j then an approximate estimator must be devised, which does not appear difficult. One may then wish to draw contour maps of the scalar field A(x,y), on the assumption that attractivity is a continuous variable. Solving for distances in this model leads to

$$d_{ij}^{\alpha} = k \frac{P_i P_j}{M_{ij}} \left(r + r \frac{M_{ij} - M_{ji}}{M_{ij} + M_{ji}} \right) ,$$

where r = a/b. This is remarkably similar to the equations obtained earlier.

A few interaction models have now been examined, and manipulated algebraically under varying assumptions. The reader will hopefully find it fruitful to extend, and improve, these results. The class of hierarchical models which also might be used to approach interaction tables has been completely neglected here [14-16].

From a mathematical point of view it is an elementary theorem of matrix algebra that every asymmetric matrix can be written as the sum of a skew-symmetric matrix and a symmetric matrix. This unique decomposition is given by

$$M^{+}_{ij} = (M_{ij} + M_{ji})/2 = M^{+}_{ji}$$

and

$$M^{-}_{ij} = (M_{ij} - M_{ji})/2 = -M^{-}_{ij}$$

where M^{+} denotes the symmetrical, and M^{-} the skew-symmetrical, portion of the table. If the usual social gravity model is written as $M_{ij} = M_{ji} = k\ P_i\ P_j\ /\ d^{\alpha}_{ij}$, then this implies that $d^{\alpha}_{ij} = 2kP_iP_j/M^{+}_{ij}$, taking the symmetry literally. If this is set equal to the comparable value

$$d_{ij}^{\alpha} = k \frac{P_i P_j}{M_{ij}} \left((r + r \frac{M_{ij} - M_{ji}}{M_{ij} + M_{ji}} \right)$$

obtained from the Somermeijer model, then one finds that r = 2 is required for the two results to be consistent. The similarity of the answers obtained under differing assumptions encourages one to believe in the robustness of the results. The recurrent appearance of the difference of the interactions in the two directions divided by their sum, M^{-}/M^{+} the relative net interaction, is particularly striking. This quantity first appeared as being proportional to the **c**'s introduced as currents.

$\mathbf{C}_{ij}$ was assumed to be the component of a current (or wind) flowing from i to j and which made interaction easier in that direction. If the locations, in geographical two-space, of the positions of i and j are known (or estimated using the trilateration procedure from d_{ij}, as defined by one of the above equations), then we can draw a small vector at i towards j of length ½$\mathbf{c}_{ij}$.The direction at i away from j is used when $\mathbf{c}_{ij}$ is negative. Since $\mathbf{c}_{ij} = -\mathbf{c}_{ji}$, one half of the vector magnitude is assigned to each of the points i and j. This happens also to make r = 2. It may be appropriate to weight the vectors by a quantity proportional to $\exp(-d_{ij})$ but this is a side issue. Analytically the calculation of the vector components is a simple trigonometric computation if the latitude and longitude coordinates of i and j are known. Doing this for all directions that interact with each point, i.e., for each i performing the computation for all j, one obtains a cluster of vectors at each point. The resultant vector sum gives an estimate of the wind field at that point. After the summation has been performed for all points, a vector field $\mathbf{c}(x,y)$ can be assumed to have been defined for all x,y in the region of observations.

From Vectors to Potentials

Every scalar field a(x,y) has associated with it a vector field, grad a. The converse, however, is not true. Nevertheless every vector field can be written as the sum of the gradient of a scalar field plus an additional vector field. These two parts are referred to as the scalar potential and the vector potential, respectively. If the second field is everywhere zero, then, and only then, the original vector field is the gradient of some scalar field. In the latter case one should be able to recover this scalar potential by reversing the gradient operation, i.e., by integration. In the present instance it is necessary to decide whether the finite set of numbers, making up the vector field and obtained from the empirical interaction table, can be considered an exact differential [17]. The idea here is that a wind implies a potential function (the attractiveness), and we would like to infer this potential from the wind.

The observed vector field must be decomposed into divergence- and curl-free parts, and the scalar and vector, potentials can then be calculated as follows [18, 19]. Recall that, for any vector field $\mathbf{c}(x,y)$, the vector identities curl grad = 0 and div curl = 0 both hold. Write $\mathbf{c}$ = grad a + curl $\mathbf{v}$ or $\mathbf{c}$ = a curl free part plus a divergence free part, and then apply the divergence operator, to obtain div $\mathbf{c}$ = div grad a + 0. But div grad = $\nabla^2 = \partial c/\partial x + \partial c/\partial y$. Thus, by calculating the divergence of our vector field we observe that the scalar field a can be obtained from Poisson's equation $\nabla^2 a = \partial c/\partial x + \partial c/\partial y$ which is solvable by known methods to obtain an estimate of the scalar potential [20].

The first part of the problem is thus resolved. Now apply the curl operator to the original equation, obtaining curl $\mathbf{c}$ = 0. + curl curl $\mathbf{v}$. If curl $\mathbf{v}$ is a vector in the x,y plane then $\mathbf{v}$ and curl curl $\mathbf{v}$ are perpendicular to it, and thus $\mathbf{c}$ has no components, nor any variation, normal to the x,y plane. Therefore, curl $\mathbf{c} = k\,(\partial c/\partial x - \partial c/\partial y)$, and analogously for curl curl $\mathbf{v}$. Thus $\nabla^2\mathbf{v} = \partial c/\partial x - \partial c/\partial y$, i.e., Poisson's equation is again to be solved, this time for $\mathbf{v} = kv$. It is necessary to solve these equations by finite difference methods at an irregular scatter of points in two dimensions.

Assuming for the moment that this can be done, there remains the problem of interpretation. The scalar field a is readily viewed as a "pressure" which induces the flow. I would expect this to be the larger of the two components of the observed field. The most reasonable interpretation that I can see for the vector potential is that of a spatial impedance. In other words, there is a mismatch or inconsistency between the flow field and the forcing function. This can perhaps be thought of as a viscosity, or as an unexplained component in the statistical sense of a residual. It is easy, too easy, to invent explanations for the mismatch indicated by the vector potential. But suppose that one has observations (i.e., interaction tables) for two, or more, time periods. In this dynamic situation one would like to consider the existence of lagged potentials. One would in fact like to calculate these from the data [21]. Ultimately one would like to consider policy changes which could be used to modify the flow field. One foresees a spatially continuous, temporally dynamic input-output scheme [22]

Other questions come to mind readily. If there is a flow of interactions in one direction, for example, must these be balanced by a counterflow of some other quantity in order to close the system: money when products flow, or decongestion (negative population density) when people move? The interpretation of interaction asymmetries as being induced by a “wind” may also be reversed. Recall that in Hägerstrand-type Monte Carlo diffusion studies [23] the mean information field is usually symmetrical. But diffusion may also take place in a field of winds, and this is a natural way of obtaining asymmetrical information fields [24]. Thus explicitly introducing these “winds” may improve the accuracy of predictions obtained from spatial interaction models. Included here are models of the gravity type as modified in this presentation arid widely used in practice, as well as others not treated here.

Examples

Several sets of data have been examined in order to clarify the foregoing concepts. The primary illustration uses a 48 by 48 table of state to state college attendance [25]. The accompanying map, Figure 1, indicates by a plus sign those states which are sources (net exporters) of students; a minus sign indicates student importers, with the area of the symbol proportional to the volume of import or export. Exportation here means that college-age residents of a state go to another state for their college education. One could have reversed the interpretation to say that some states export education to non-residents; but the students actually move, and this has been labeled an exportation. New Jersey was, in 1968, by far the largest exporting state by this criterion, with 90,000 students studying in other states. This state, compared to its educational facilities, had a surplus of educationally motivated residents[1]. The computed vector field is shown in the next illustration, Figure 2. The vectors have been positioned at the approximate center of gravity of their resolution elements (states), which introduces an error of less than one half of the resolution element. The components of the vector are computed as

$$\vec{c}_i = \frac{1}{n-1} \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{M_{ij} - M_{ji}}{M_{ij} + M_{ji}} \cdot \frac{1}{d_{ij}} \cdot [(X_j - X_i), (Y_j - Y_i)],$$

where

$$d_{ij}^2 = (X_j - X_i)^2 + (Y_j - Y_i)^2.$$

The vector **c** is plotted at the map location (X_i , Y_i), after an arbitrary scaling appropriate to the particular map, using plane coordinates on a local map projection [26]. The foregoing formula is derived by simple trigonometry from the interpretation of $\mathbf{c}_i$ as the components of a vector bound at i, directed from i to j.

The next figure, Figure 3, shows a spatially weighted version of the same data, computed as

$$\vec{c}_i = \frac{1}{n-1} \sum_{\substack{j=1 \\ j \neq i}}^{n} W_j \, \frac{M_{ij} - M_{ji}}{M_{ij} + M_{ji}} \cdot \frac{1}{d_{ij}} \cdot [(X_j - X_i), (Y_j - Y_i)]$$

with

$$W_j = \frac{\dfrac{1}{d_{ij}}}{\sum\limits_{\substack{j=1 \\ j \neq i}}^{n} \dfrac{1}{d_{ij}}} = \frac{1}{d_{ij} \sum\limits_{\substack{j=1 \\ j \neq i}}^{n} \dfrac{1}{d_{ij}}} .$$

The motivation here is that local influences should carry more weight than distant ones. But the difference between the figures is comparatively minor, and this seems to be the case for other examples which have been computed. As a contrast it is additionally useful to display a map, Figure 4, constructed from a 48 by 48 table of non-negative random numbers. The pattern computed from this random table lacks structure, the degree of spatial auto-correlation within the vector field being small. This does not appear to be the case with maps computed from real interaction data.

Using the forty-eight vectors located at state centroids one can estimate values at a regular spatial lattice to obtain a somewhat more legible picture of the vector field, Figure 5. In the present instance this spatial interpolation captures better than 85 per cent of the pattern of the irregular field. The positioning of the lattice is rather crude, with several vectors falling outside the United States; this is of course not meaningful since there are no observations in that region. But the main purpose of the assignment of values to the lattice positions is to facilitate the solution of Poisson’s equation by finite difference methods. While this could in principle be done for data arrayed at state centroids, the additional computer programming effort did not seem warranted at this stage. A program that iterates by finite differences for data at a regular array of points seemed adequate as an initial attempt. Both the scalar and vector potential functions have been calculated by such a program, which also plots contour maps of these functions. The boundary conditions are that no movement can cross the border of the region, since the system of student flows is closed as far as these data are concerned. Alternate specifications for the boundary yield different solutions to the Dirichlet problem. Considering the field of vectors as a velocity field suggests an alternate boundary specification.

One might argue that the potentials should have been computed from a field based on net flows, rather than on relative net flows. But if one examines the relation obtained from the Somermeijer model

$$Q_{ij} = A_j - A_i = \frac{a}{b}\left(\frac{M_{ji} - M_{ij}}{M_{ij} + M_{ji}}\right),$$

and postulates the existence of a scalar function **A(x,y),** then a finite approximation to the directional derivative based on observations at i and j would be

But the directional derivative in any direction is the component of the gradient in that direction,

$$\frac{\partial A}{\partial S_\theta} \simeq \frac{A_j - A_i}{d_{ij}}.$$

and thus the gradient is a linear combination of directional derivatives. The formula used to compute the components of the vector field is just such a linear combination.[2] Thus the Somermeijer model suggests that the relative net interaction is the more appropriate expression.

Superimposition of the map of the vector field on that of the scalar potential, Figure 6, shows the degree to which the field corresponds to the gradient of the scalar function. Perfect agreement would hold if the vectors were everywhere orthogonal to the contours. This of course would imply that curl **v** = 0, as previously noted. Beckmann [27, 28], several years ago, asserted that flows would be proportional to gradients of some scalar function, and a crude test of this assertion is now provided. The map of the vector potential calculated as a residuum is also shown, Figure 7.

The most natural hypothesis for the movement of students is to postulate that the computed scalar potential, Figure 6, is related to the departure, by state, from the national educational expenditure, per capita. This hypothesis can be approached in several ways. The tack used here is to compute an expected student-flow field. An estimate of the 1968 per capita expenditure for higher educational plant, by state, is readily available [29]. On this basis, letting A_i represent the educational expenditure in state i and d_{ij}, the distance (km) of state i from state j, the quantity

$$\frac{A_j - A_i}{d_{ij}} \simeq \frac{\partial A}{\partial S_\theta}$$

was formed at each i as a vector in the direction θ, towards j. The resultant of the several vectors at each point was then used to obtain a "continuous" vector field, Figure 8. This field, if our hypothesis has any merit, should agree with the one calculated independently from the student flow table. But the map which is offered in evidence only modestly resembles the field of student migrations. The correlation (R^2) between a numerical estimate of the attractivity (scalar potential) and this variable (educational expenditure) is only 70 per cent. Thus, on two counts, one is forced to conjecture additional relationships. The point is that the map of the scalar potential may of itself suggest likely relationships, particularly in situations more obscure than student flows. A flow field computed from empirical data might also be compared to a flow field obtained from an interaction table generated by some model; the differences in the fields may then again lead to insights.

The next two maps, Figures 9 and 10, are of related cohorts, and are based on tables [30] showing the number of persons receiving a Bachelor's degree in state i who received a Ph.D. degree in state j, and the number of persons receiving their Ph.D. degree in state i who took their first post-Ph.D.-degree employment in state j. The latter might be considered as a type of "brain drain." All three sets of data are best considered together, even though they do not trace the paths of specific individuals. A number of interesting investigations are suggested by these tables, including of course the structure of the transition probability matrices. A somewhat comparable effect is obtained by comparing the three vector fields.

It should be clear that the technique described can be applied to a variety of data. Interregional commodity flow tables collected for input/output analyses constitute one accessible source [31, 32]. Migration tables are also of interest [33]. Here it becomes obvious that, since people move for different reasons, a potential function calculated from an asymmetric migration table must be the sum of individual potential functions. Thus it would be advisable to disaggregate such tables by age, by occupational group, and perhaps by other categories. As related examples, Figure 11 shows the 1939 commuting pattern (journey-to-work) for the city of Munich, and another, Figure 12, shows the December 1970 commuting field in Belgium [34]. The pattern of commuting in Munich is very clear, and is as one expects. Belgium, due to the larger areal coverage, has several centers of employment and presents a much more mixed pattern. Figure 13 represents tourist travel within that portion of Europe for which there are data and this again shows a rather coherent pattern [35].

The next two maps, Figures 14 and 15, show the flow of information. In the one case the field is computed from a 17 by 17 table of the maximum number of telephone calls from one exchange to another in the city of Zürich in 1970 [36]. One is tempted to place a vortex just west of Parade Platz, where important banks are located. The second figure shows business contacts in Sweden, estimated as an origin-destination table obtained from a sample of airline passengers [37]. The major Swedish cities stand out clearly.

As a final, non-geographical application, the asymmetry between psychological journal citations has been analyzed. The first step is to position the journals spatially. This could be done using one of the formulae derived earlier, or by treating the elements of M^{+}_{ij} as similarities. In the present instance the data have already been located in a two-space by a multidimensional scaling algorithm and it only remains to use the asymmetries to assign vectors to these data points [38]. Interestingly the flow, Figure 16, is from the experimental journal to the clinical journal, but the vector field does not match the gradient of the scalar potential very well. The further task of interpreting this description of the psychological literature is best left to that profession, and the sample size is small. But the procedure can be applied to any asymmetrical table.

Caveats

One of the reasons for including so many examples in this paper is because the technique works with any set of data. One can take any asymmetrical table and from it compute a vector map. The procedure cannot fail, which immediately makes it somewhat suspect. Furthermore, the potentials can never actually be observed, but can only be deduced from their consequences. Thus one can ask whether the potentials should be computed from the relative net flows (as has been done here), or directly from the net flows (which intuitively seems more reasonable), or from a table adjusted so that the rows sum to unity (as in a probability transition table), or in some other manner. The results will in each case differ, albeit only slightly. The plausibility of the several example fields, the relation to the Somermeijer model, and the lack of pattern in a vector field computed from a random table, all combine to suggest (perhaps more fortuitously than algebraically) that the technique has some merit. Thus, several vector field patterns, computed from a number of empirical interaction tables[3] have been examined. In each case these patterns have seemed plausible. This means that one could reasonably expect the observed patterns on the basis of a priori knowledge of the phenomena analyzed. The conclusion is therefore that the method is valid. But the existence of a potential function is already contained in the assumption of gravity model, and the domain of validity of such models is restricted to macroscopic effects. It has furthermore been convenient to consider space, fields, and potentials as continuous entities. The proximate nature of these assumptions should be kept in mind in any applications. On the other hand the method appears to be insensitive to alternate regionalizations of the data, and yields more detailed results the finer the geographical resolution of the initial interaction table.

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[1]If one separates the 48 by 48 college attendance table into symmetrical and skew-symmetrical parts, then the latter portion contains 24 per cent of the total variance.

[2]In principle the gradient can be computed from the basis formed by two independent directional derivatives. But which two? It is easily possible that. one can derive a better approximation to the gradient than the averaged value used here. The problem is one of finding the tangent plane to a surface from finite measurements. The essential argument of this essay is not dependent on the particular approximating equation used, although the point requires further clarification.

[3]Whose accuracy, incidentally, is not known

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REFERENCES

1. W. Tobler, H. Mielke and T. Detwyler, Geobotanical Distance Between New Zealand and Neighboring Islands, *Bioscience, 20:9*, pp. 537-542, 1970.
2. W. Tobler and S. Wineberg, A Cappadocian Speculation, *Nature,* 231:5297, pp. 39-42, May

7, 1971.
3. P. Wolf, Horizontal Position Adjustment, *Surveying and Mapping, 29:4,* pp. 635-644, 1969.
4. R. Shepard, The Analysis of Proximities, *Psychometrika, 27,* pp. 219-246, 1962.
5. J. Lingoes, *The Guttman-Lingoes Non Metric Program Series,* Mathesis Press, Ann Arbor, pp. 81-111, 1973.
6. P. Schwind, *Migration and Regional Development in the United States: 1950-1960,* University of Chicago, Department of Geography, Chicago, Research Paper 1933, 1971.
7. W. Borejko, Study of the Effectiveness of Migrations, *Geographic Polonica, 14,* pp. 305.312, 1968.
8. A. Wilson, *Urban and Regional Models in Geography and Planning,* J. Wiley, New York, 1974.
9. I. Lowry, *Migration and Metropolitan Growth,* Chandler, San Francisco, 1966.
10. M. Cordey-Hayes, On the Feasibility of Simulating the Relationship Between Regional Imbalance and City Growth, in *Space-Time Concepts in Urban and Regional Models,* E. Cripps, (ed.), Pion, London, p. 193, 1974.
11. J. H. Ross, *A Measure of Site Attraction,* Ph.D. thesis, Department of Geography, University of Western Ontario, London, Canada, p. 27 et seq, 1972.
12. F. Cesario, A Generalized Trip Distribution Model, *Journal of Regional Science, 13:2,* pp. 233-247, 1973.
13. W. Somermeijer, Multi-Polar Human Flow Models, *Papers*, Regional Science Assa., *26*, pp. 131-144, 1971.
14. M. Dacey and J. Nystuen, A Graph Theory Interpretation of Nodal Regions, *Papers,* Regional Science Assn., 7, pp. 29-42, 1961.
15. M. Harvey and J. Auwerter, Derivation and Decomposition of Hierarchies From Interaction Data, Discussion Paper 31, Department of Geography, Ohio State University, Columbus, Ohio, 1973.
16. P. B. Slater, College Student Mobility in the United States in Fall 1968: Graph-Theoretic Structuring of Transaction Flows, *Proceedings,* Social Statistics Section, Am. Stat. Assn., 9:3, pp. 440-446, 1974.
17. S. Salas and E. Hille, *Calculus,* Toronto, Xerox College Publishing, Part II, pp. 659.666, 1969.
18. H. N. Pollock, On the Separation of an Arbitrary Two Component Vector Into Divergence Free and Curl Free Parts, and the Determination of the Scalar and Vector Potentials, mimeo, Ann Arbor, 2 Pp., 1973.
19. H. M. Schey, *Div, Grad, Curl, and All That*, Norton, New York, 1973.
20. B. Carnahan, H. Luther and J. Wilkes, *Applied Numerical Methods,* J. Wiley, New York, pp. 429.530, 1969.
21. R. McQuistan, *Scalar and Vector Fields,* J. Wiley, New York, pp. 292-305, 1965.
22. W. Leontief and A. Strout, Multiregional Input Output Analysis, in *Structural Interdependence and Economic Development*, T. Barna, (ed.), Macmillan, London, pp. 119-150, 1963.

23. T. Hägerstrand, A Monte Carlo Approach to Diffusion, in *Spatial Analysis,* B. Berry and D. Marble, (eds.), Prentice Hall, New York, 1968.
24. F. Pasquill, *Atmospheric Diffusion: the Dispersion of Windborne Material From Industrial and Other Sources,* Van Nostrand, New York, 1962.
25. National Center for Educational Statistics, *Residence and Migration of College Students: Basic State-to-State Matrix Tables,* United States Department of Health, Education, and Welfare, Government Printing Office, Washington, D.C., 1970.
26. W.Tobler, Local Map Projections, *The American Cartographer,* 1:1, pp. 51-62, 1974.
27. M. Beckmann, On the Equilibrium Distribution of Population in Space, *Bulletin of Mathematical Biophysics, 19,* pp. 81-90, 1957.
28. W. Warntz, *Macro Geography and Income Fronts,* Regional Science Research Institute, Philadelphia, 1965.
29. Department of Commerce, *Statistical Abstract of the United States, 1972,* G.P.O., Washington, D.C., pp. 130.132.
30. Provided by Professor W. Dekan of the University of Wisconsin, White Water, Wisconsin.
31. J. M. Rodgers, *State Estimates of Interregional Commodity Trade, 1963,* Lexington Heath Books, Cambridge, p. 447, 1973.
32. W.Tobler, *Commodity Fields,* internal paper, International Institute for Applied Systems Analysis, Laxenburg, March 1975.
33. W. Tobler, *Migration Fields*, internal paper, International Institute for Applied Systems Analysis, Laxenburg, March 1975.
34. D. Fliedner, Zyklonale Tendenzen bei Bevölkerungs und Verkehrsbewegungen in Städtischen Bereichen untersucht am Beispiel der Städte Göttingen, München, und Osnabrück, *Neues Archiv für Niedersachsen, 10:15,* , Table 2, p. 281, April 4, 1965. Belgium census commuting data provided by Mr. Frans Willekens of IIASA.
35. OECD, Nights Spent by Foreign Tourists, *Tourism Policy and International Tourism in OECD Member Countries, 1973,* OECD, Paris, pp. 95-141, 1974.
36. Data provided by Professor A. Kilchenmann, Geographisches Institut, Technische Universitit Karlsruhe, FRG.
37. G. Törnqvist, *Contact Systems and Regional Development,* Gleerup, Lund, *p.* 77, 1970.
38. C. Coombs, J. Dawes and A. Tversky, *Mathematical Psychology,* Prentice Hall, Englewood Cliffs, pp. 73-74, 1970.
39. D. Thompson, Spatial Interaction Data, *Annals,* Assn. Am. Geographers, 64:4, pp. 560-57 5.

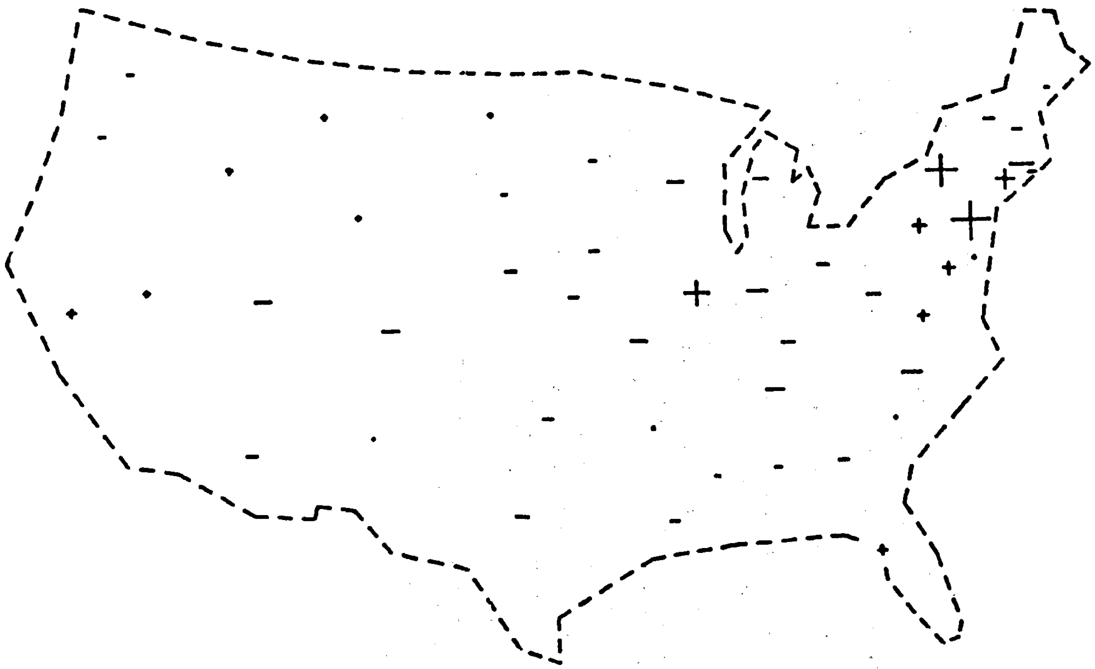


Figure 1. Exportation of students by state, 1968.

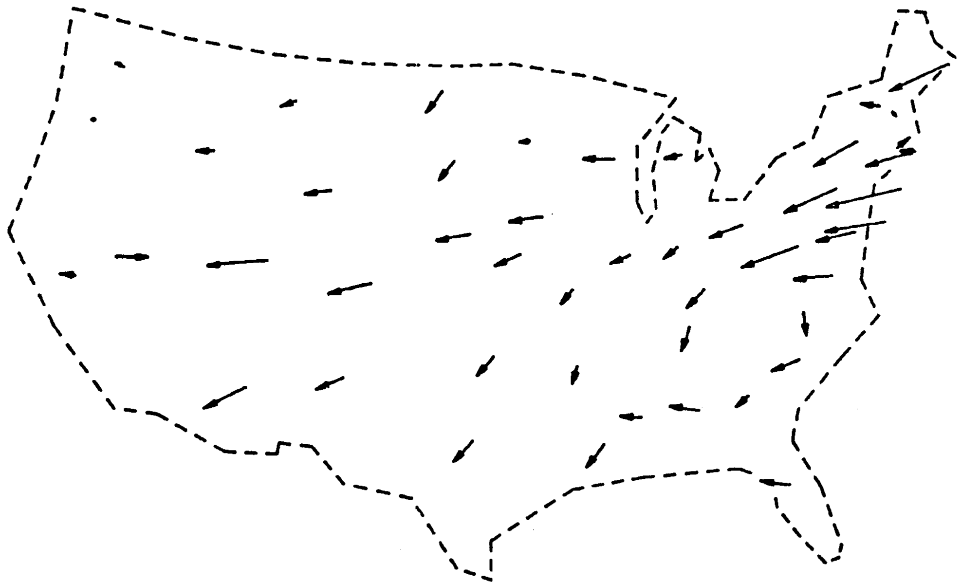


Figure 2. Field of asymmetric student flows, 1968.

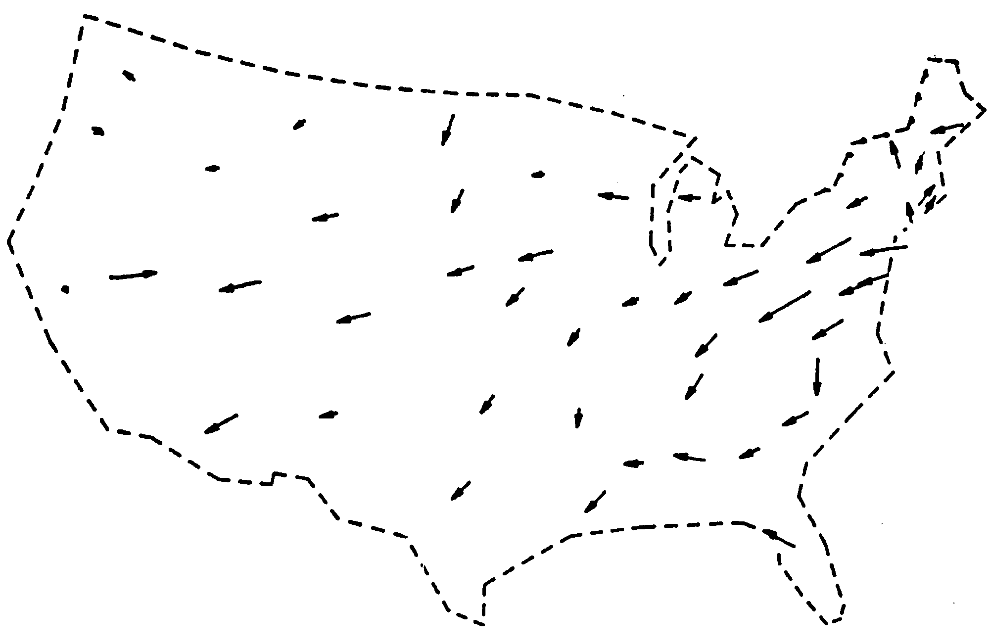


Figure 3. Weighted student flow field, 1968.

Figure 4. Flow field computed from a random interaction table.

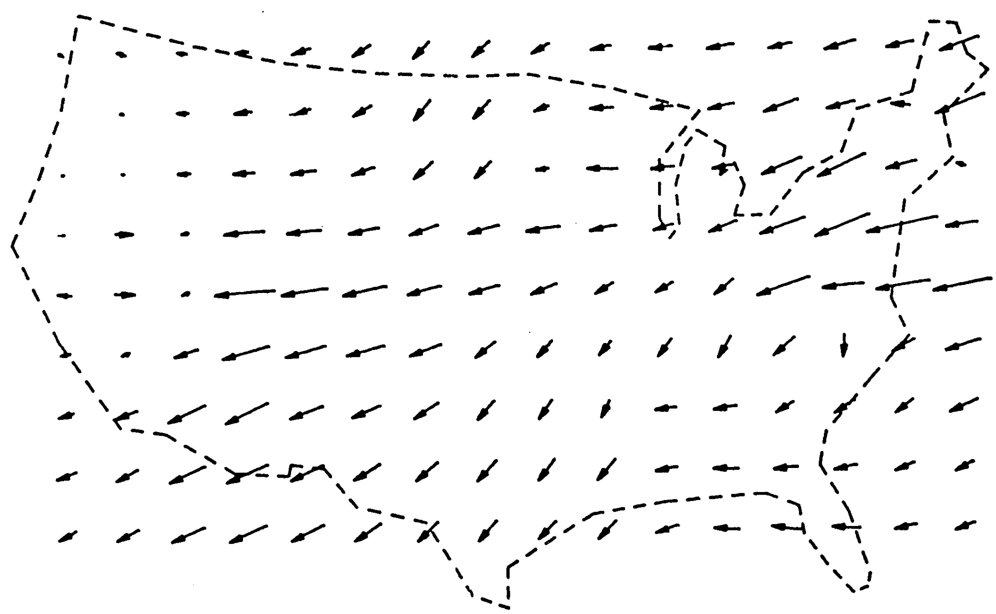


Figure 5. Interpolated student flow field, 1968.

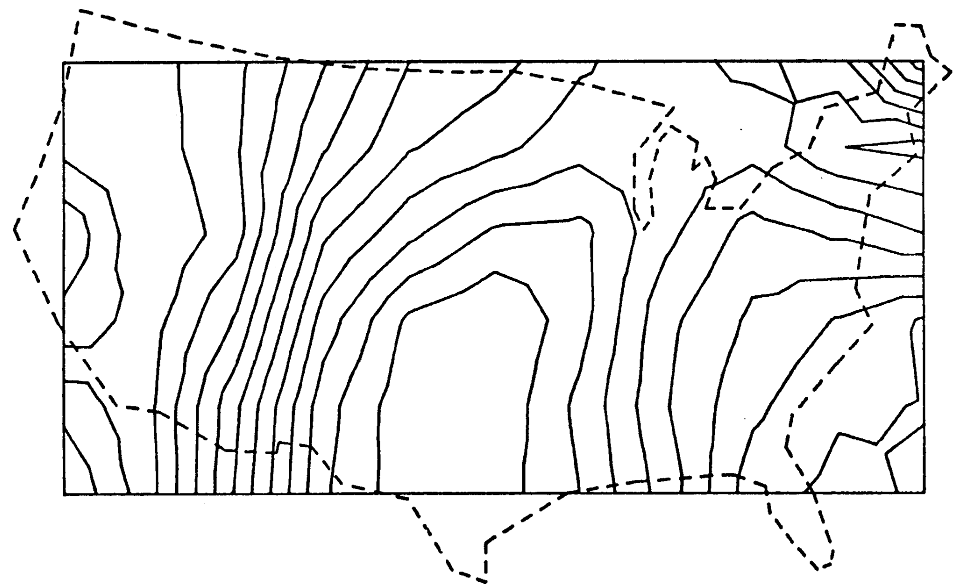


Figure 6. Scalar potential of student flows, 1968.

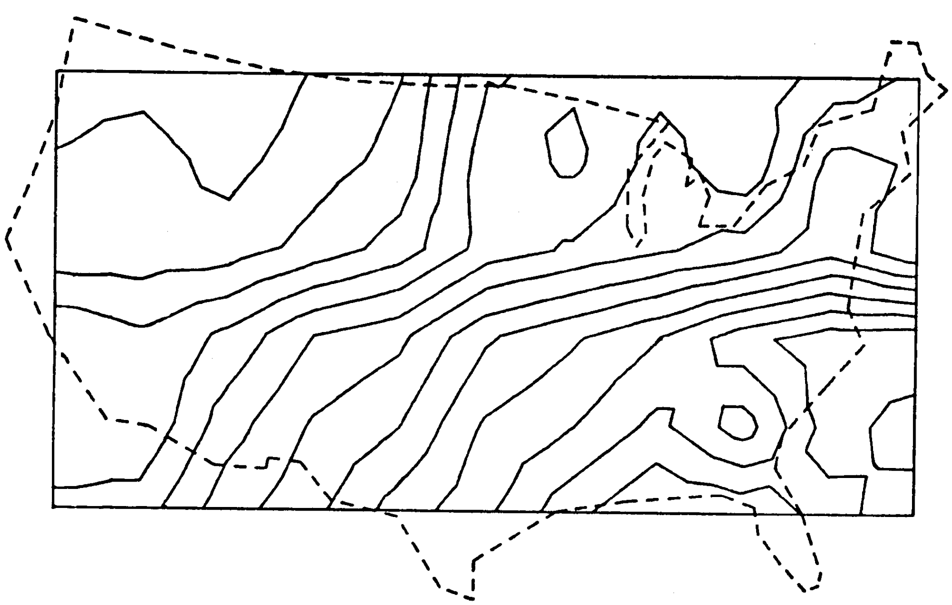


Figure 7. Vector potential of student flows, 1968.

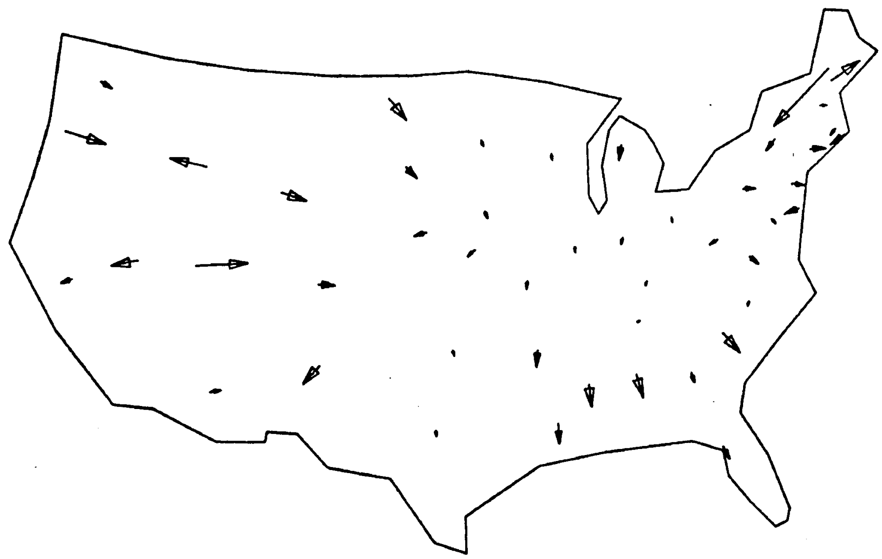


Figure 8. Gradient field of educational expenditures, 1968.

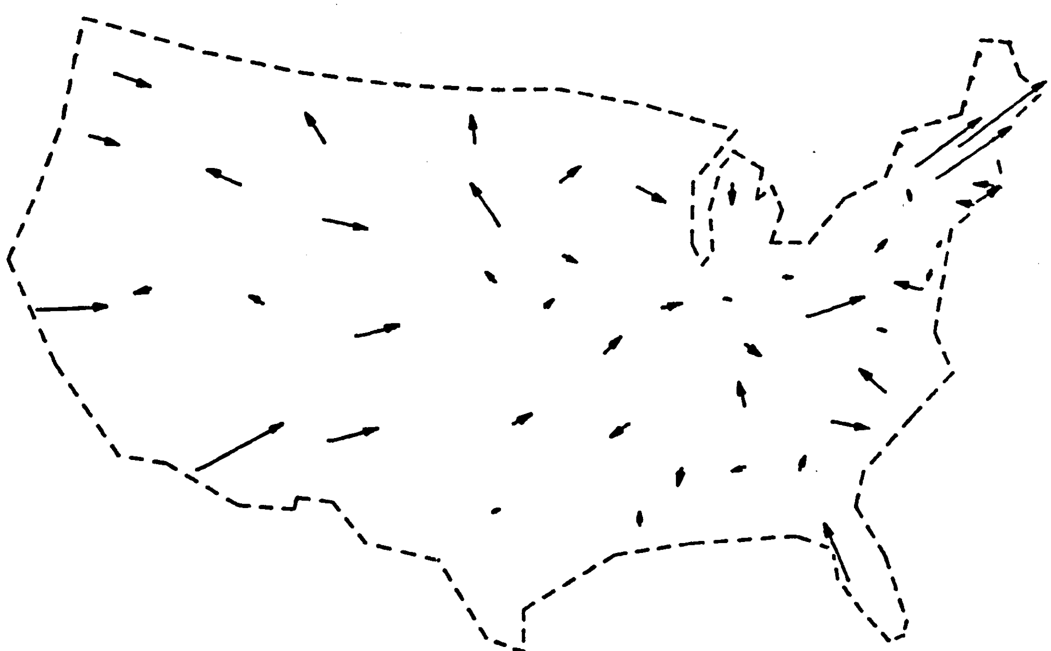


Figure 9. B.A. to Ph.D. flow field, 1968.

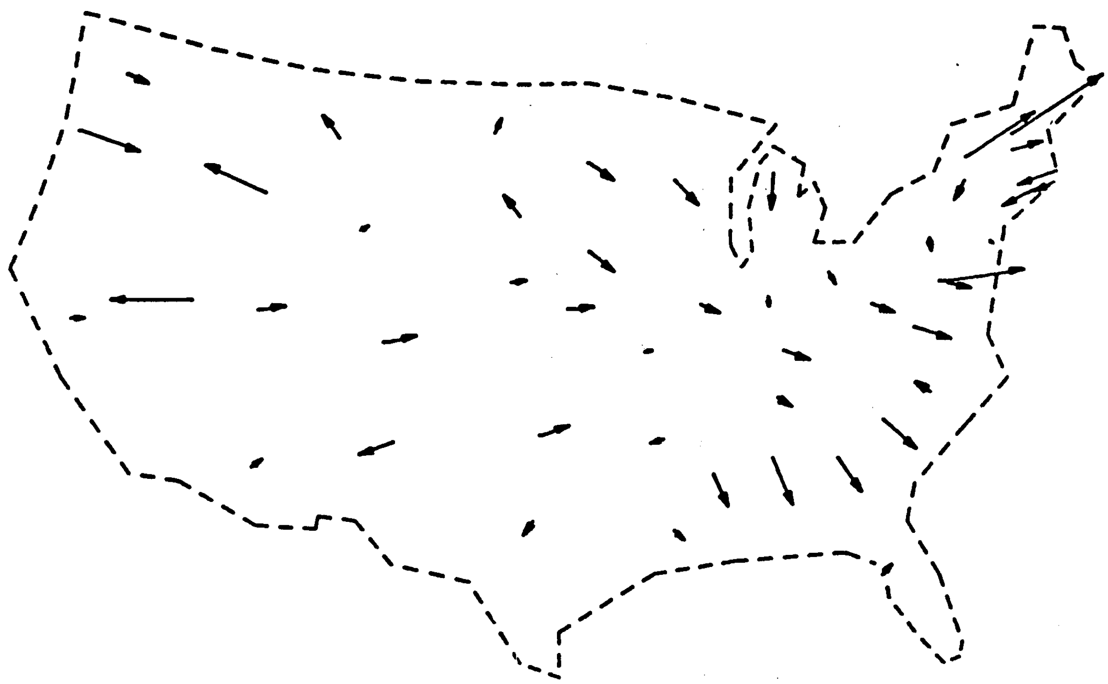


Figure 10. Ph.D. to employment flow field, 1968.

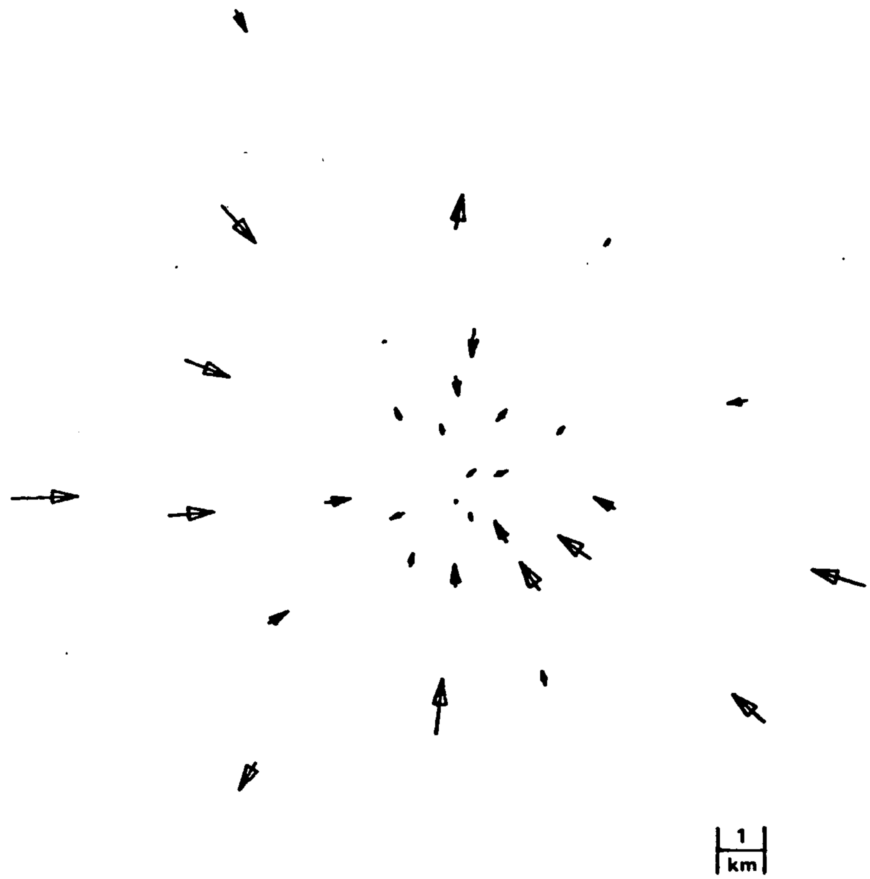


Figure 11. Commuting field for Munich, 1939.

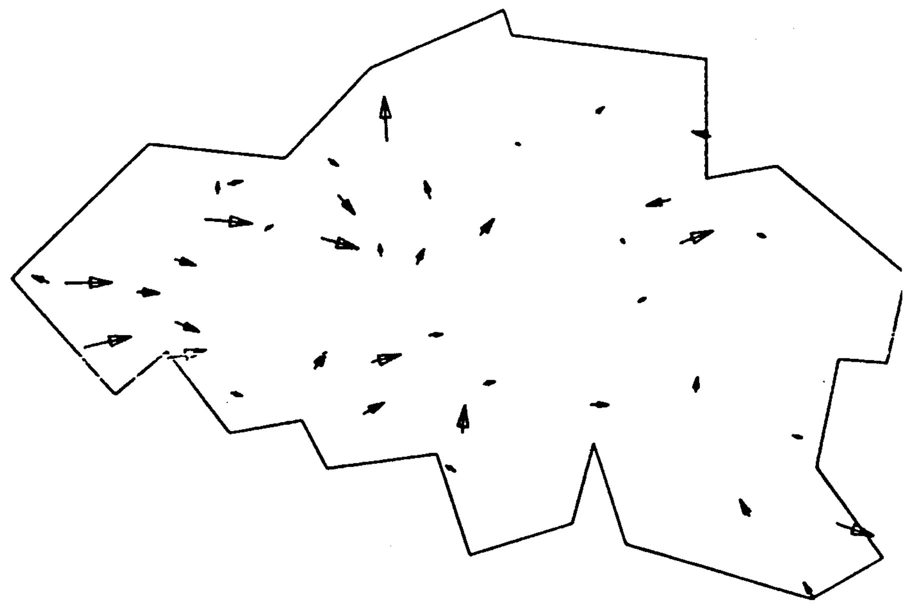


Figure 12. Commuting field for Belgium, 1970.

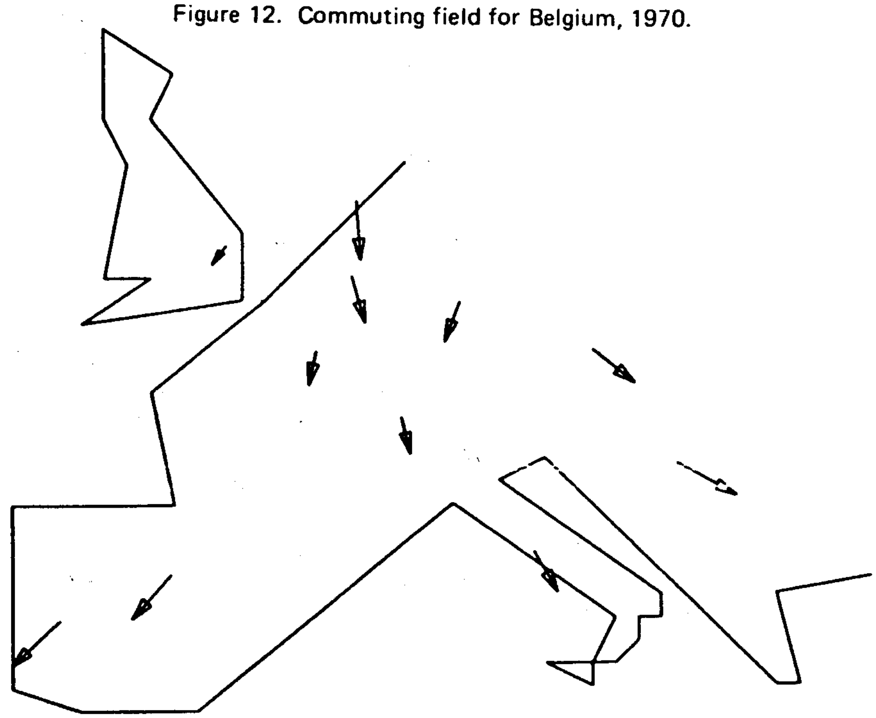


Figure 13. European tourist exchanges, 1973.

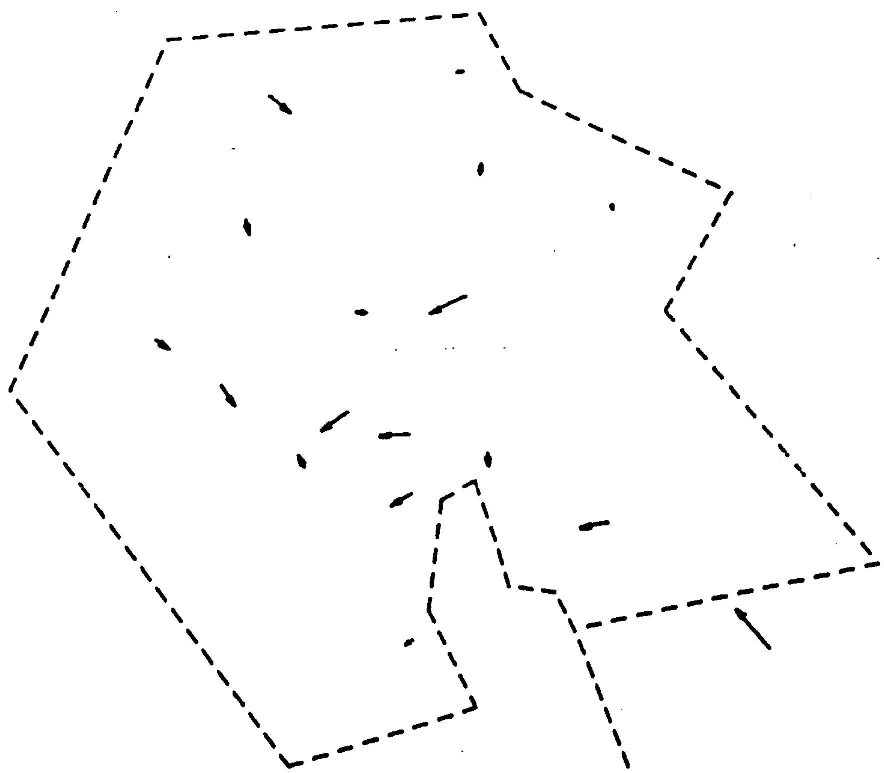


Figure 14. Telephone flow field for Zurich, 1970.

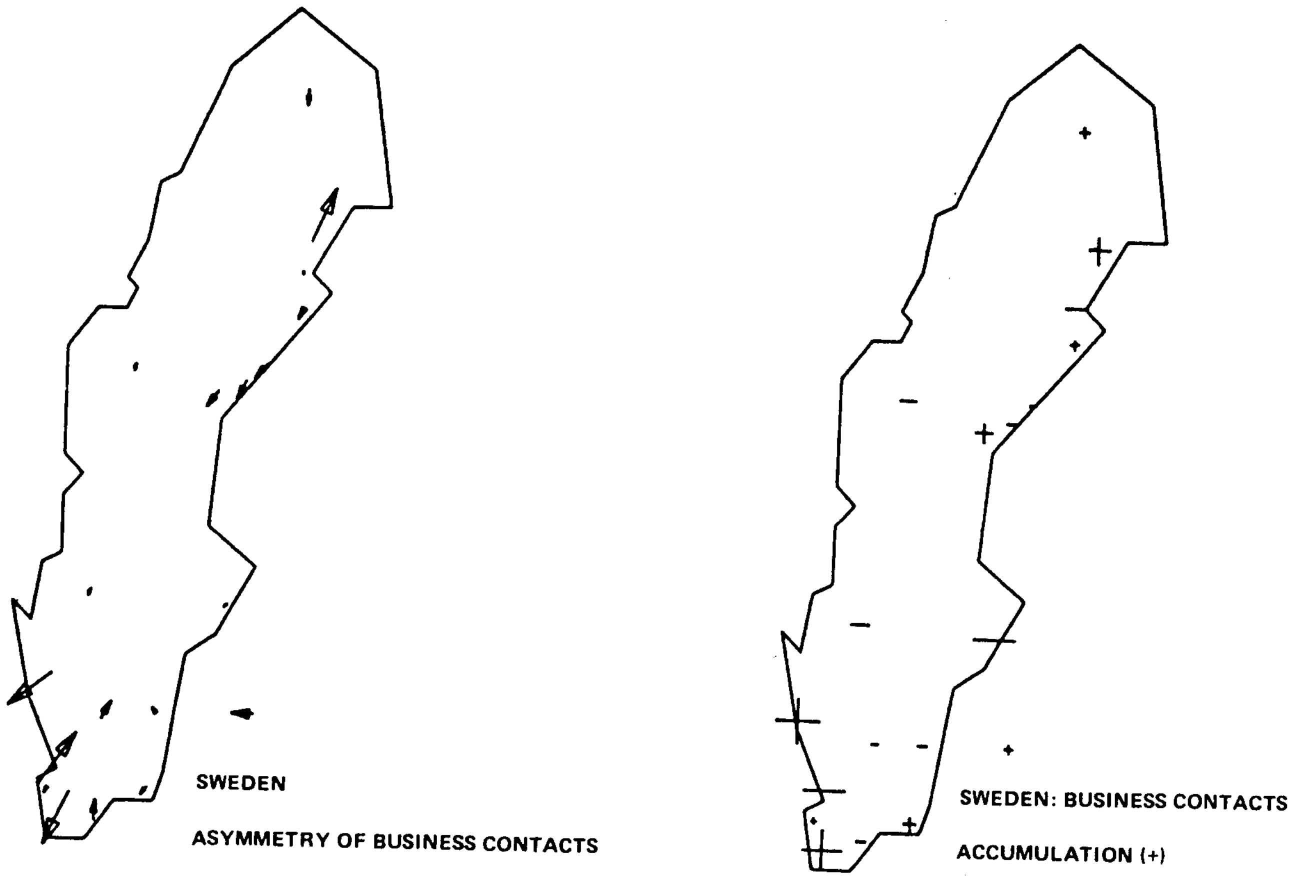


Figure 15. Business contact field for Sweden.

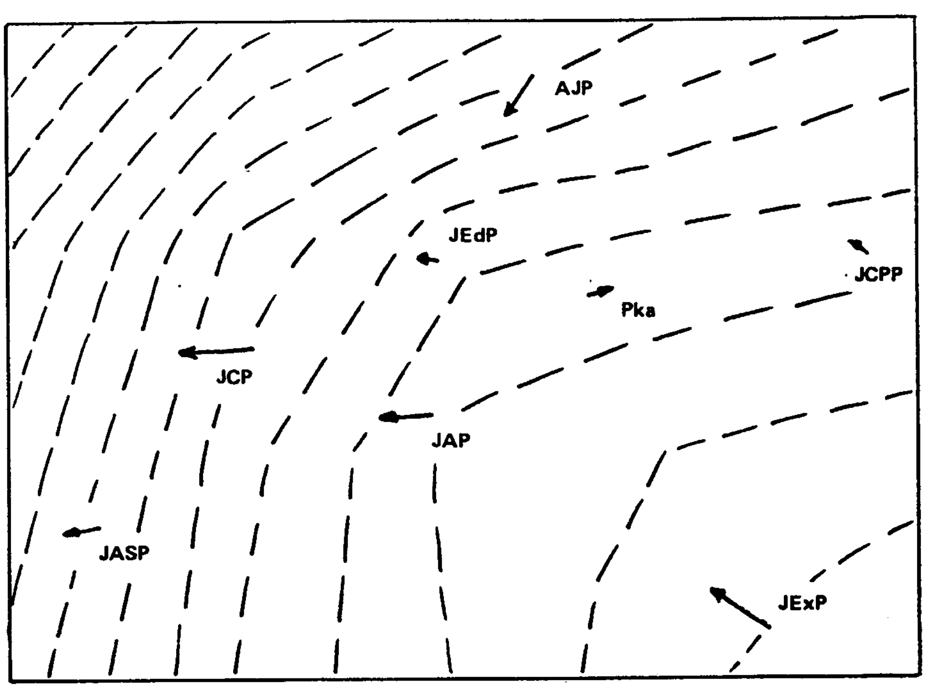
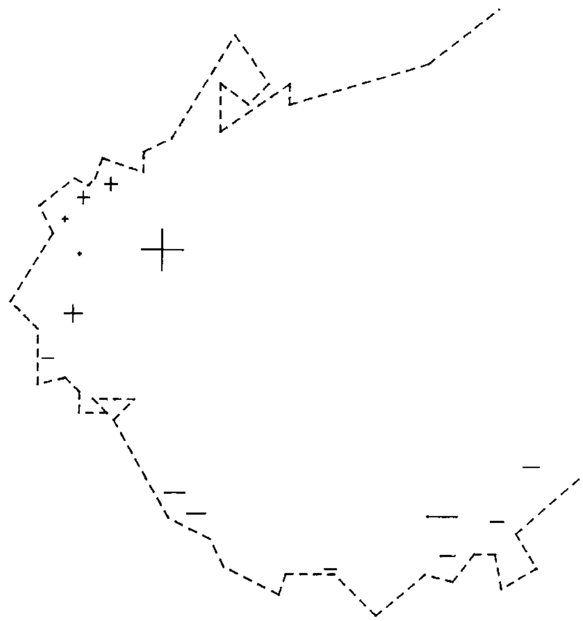
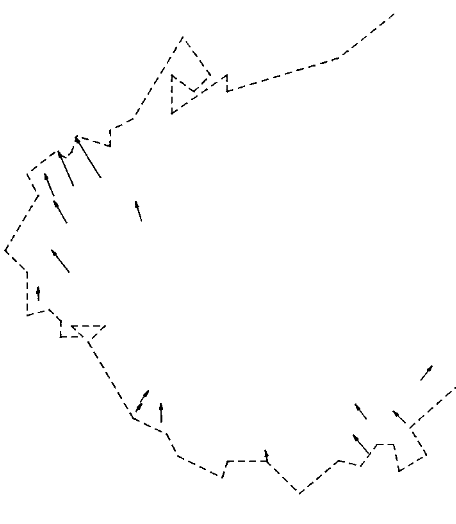


Figure 16. Flows between psychological journals.

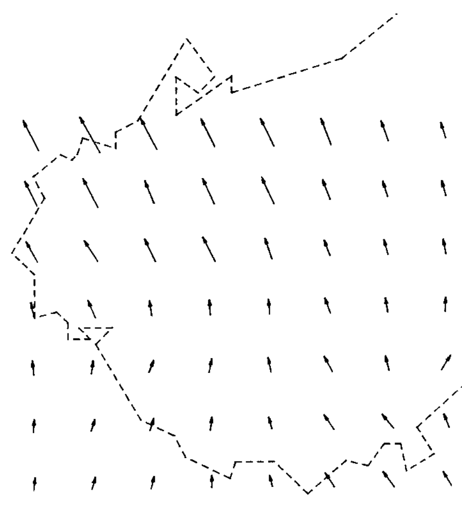
Some additional results
USSR Migration between Republics, circa 1970



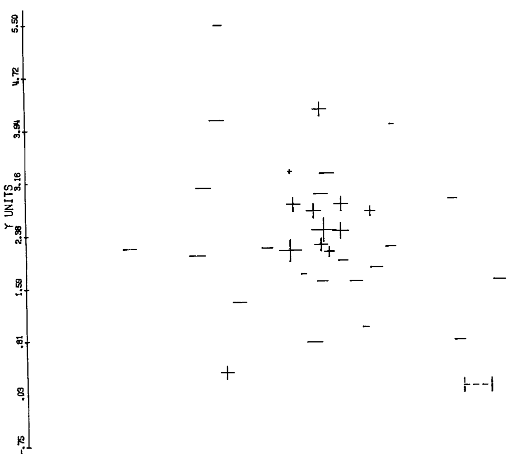
USSR MIGRATION BETWEEN REPUBLICS (CIRCA 1970)



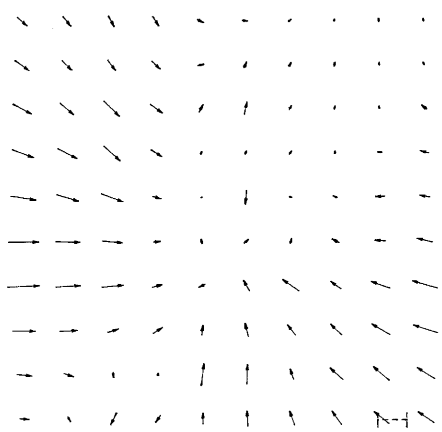
USSR MIGRATION BETWEEN REPUBLICS (CIRCA 1970)



INTERPOLATED VECTOR FIELD
USSR MIGRATION BETWEEN REPUBLICS (CIRCA 1970)



COMMUTING PATTERN IN MUNICH (1939)

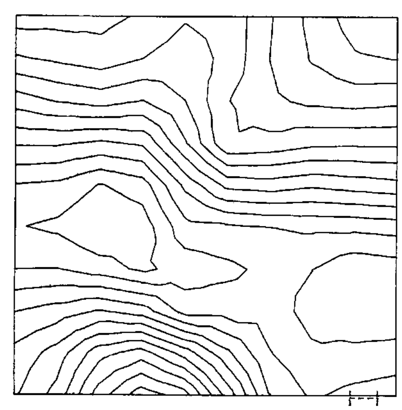


INTERPOLATED VECTOR FIELD

COMMUTING PATTERN IN MUNICH (1939)

.38 1.16 1.95 2.73 3.51 4.29 5.07 5.85 6.64

X UNITS



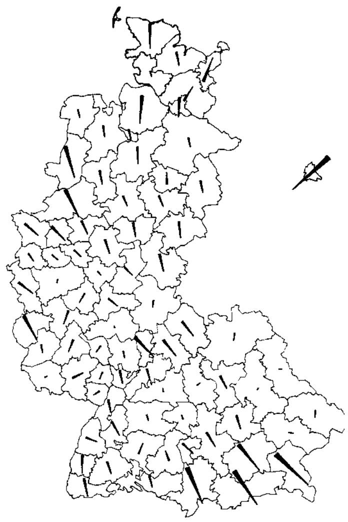
COMMUTING PATTERN IN MUNICH (1939)

THE FORCING FUNCTION

.38 1.16 1.95 2.73 3.51 4.29 5.07 5.86 6.64

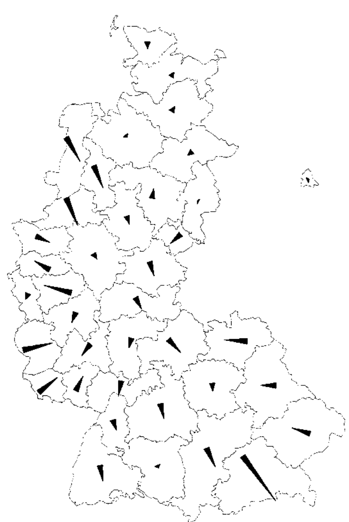
X UNITS

West Germany, Migration Winds. Maps by Wolf Rase



RICHTUNG UND STAERKE DES "WANDERUNGSWINDES"
WANDERUNGEN ZWISCHEN STATISTISCHEN RAUMEINHEITEN

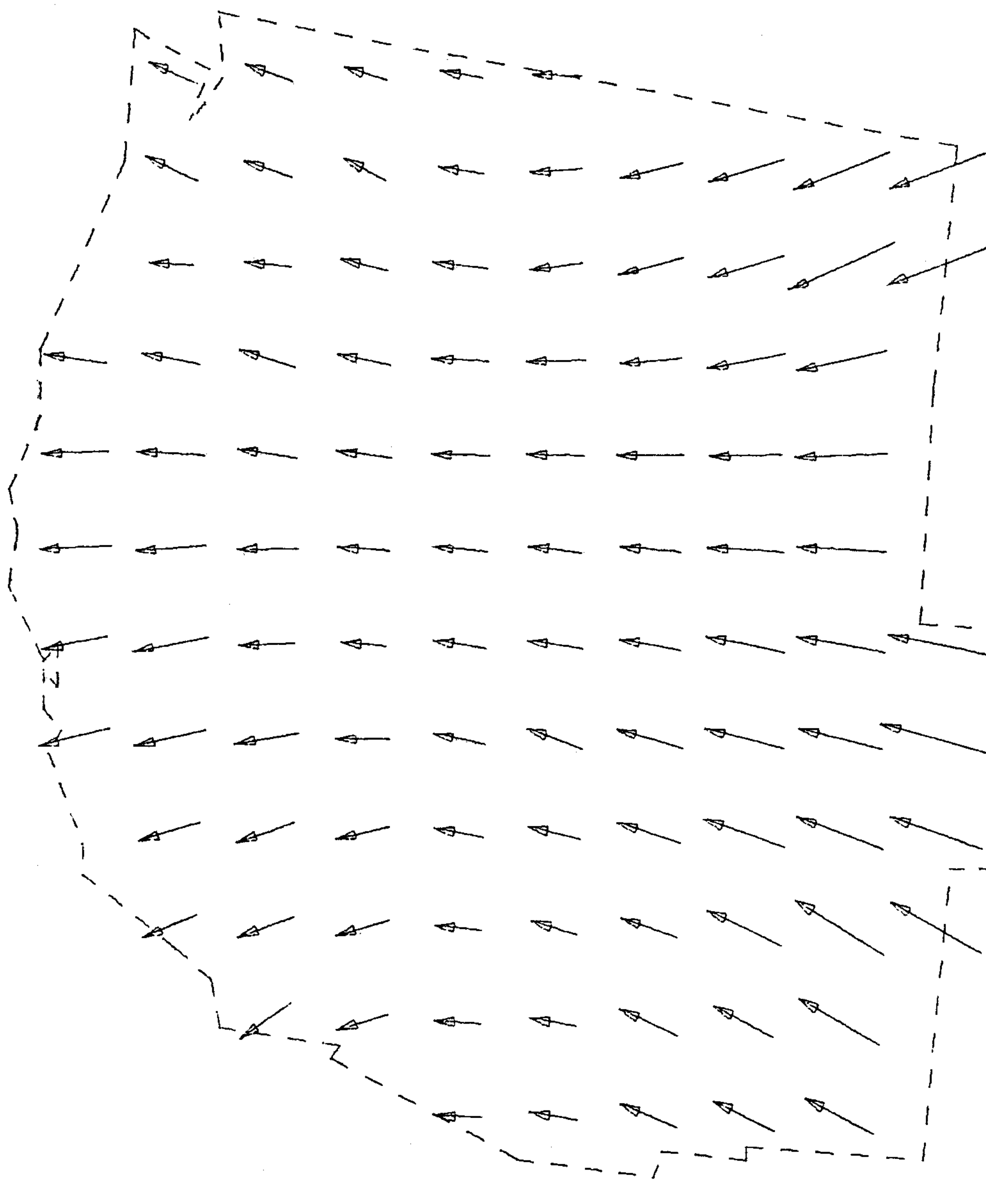
RICHTUNG UND STAERKE DES "WANDERUNGSWINDES"
WANDERUNGEN INSGESAMT ZWISCHEN RO-GEBIETSEINHEITEN



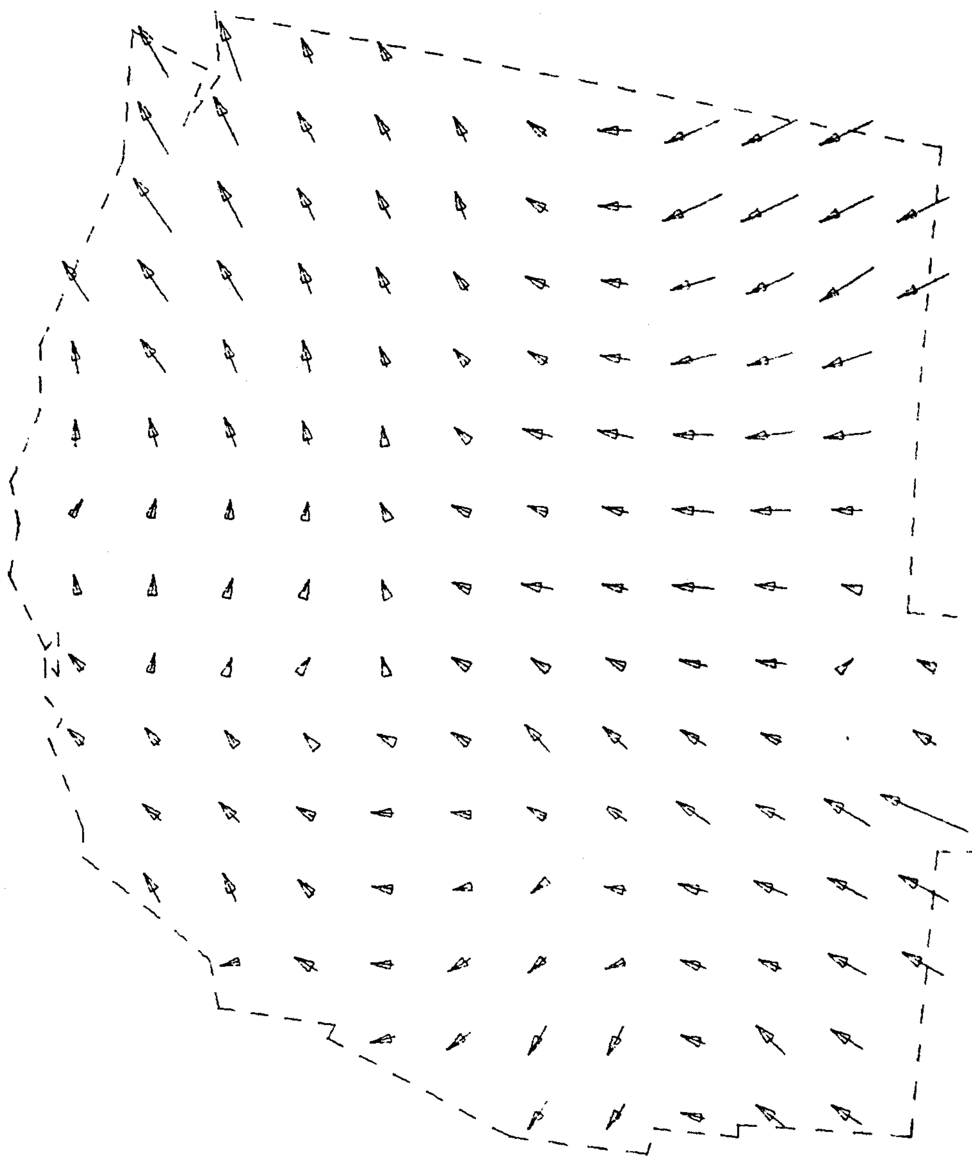
RICHTUNG UND STAERKE DES "WANDERUNGSWINDES"
ALTERSGRUPPE < 50 JAHRE

RICHTUNG UND STAERKE DES "WANDERUNGSWINDES"
ALTERSGRUPPE > 49 JAHRE

Migration in the Western United States, 1935-1940. Based on State Economic Area data.



Migration in the Western United States 1965-1970. Based on Sate Economic Area data.



POPULATION DENSITY BY COUNTY

WESTERN U.S. - 1950

LEGEND

ISOPLETH INTERVALS:

(INHABITANTS PER SQ. MI. BY COUNTIES)

5	480
10	640
20	960
40	1280
80	2560
160	5160
240	10240
320	

SCALE:

100 50 0 100 200 300 400 MI.

SOURCE: COUNTY & CITY DATA BOOK - 1952, U.S. BUREAU OF CENSUS

MARK HAUN