A CONTINUOUS TRANSFORMATION USEFUL FOR DISTRICTING[1]

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Abstract: Imagine that one could stretch a geographical map so that areas with many people would appear large, and areas with few people would appear small. If such a map could be made one would expect all voting districts to be the same size for they should contain equal numbers of people. Drawing on such maps should simplify the process of creating district boundaries. The construction of maps of the requisite type is shown to require the simultaneous solution of a pair of nonlinear partial differential equations, for which an iterative computer solution procedure has been devised. An experimental attempt to district using this method is described.

Suppose that one could stretch a geographical map so that areas containing many people would appear large, and areas containing few people would appear small. On a rubber map, for example, every person might be represented by an inked dot. We now imagine the rubber sheet to be stretched so that all the dots are at an equal distance from each other (see Ruston 1971). If such a map could be constructed, then all perfect political districts should be the same size, for they should contain equal numbers of people. Alternately, one might wish to construct district boundaries by drawing them as hexagons on such a map. These general notions are made more precise and given mathematical definition in the paragraphs that follow. The political problem of districting is related to a classical theoretical problem in the field of geography. The location-theoretic problem of positioning service facilities (schools, hospitals, stores, cities, and so forth) in a geographic area of varying population density has a structure similar to the districting problem (see Bunge 1966). This theoretical problem in geography provided the main impetus for my research, and it is reflected in the results to be demonstrated.

Assume first that the relevant population density is described as a continuous nonnegative function of position, $h(x, y)$. A small rectangle bounded by x, $x + \Delta x$, y, $y + \Delta y$ then contains $h(x, y)\, \Delta x\, \Delta y$ people. This number is to be the same as the area of a small rectangle bounded by the lines u, $u + \Delta u$, v, $v + \Delta v$ on the final diagram. The condition equation thus becomes $h(x, y)\, \Delta x\, \Delta y = \Delta u\, \Delta v$. Passing to differentials, we have $h(x, y)\, dx\, dy = du\, dv$, and, inserting the Jacobian determinant for a change of variables $h(x, y)\, dx\, dy = [(\partial u/\partial x)(\partial v/\partial y) - (\partial u/\partial y)(\partial v/\partial x)]\, dx\, dy$. The basic differential equation of the system is thus $J = h(x, y)$. This is immediately seen to be a generalization of the notion of an equal area mapping and easily extended to a sphere or ellipsoid (see Tobler

1963). We do not wish to place any restrictions, other than boundedness, on the density function h, thus ruling out some simpler solutions to this equation, as for example might be suggested if h were separable, or centrally symmetric when given in polar coordinates. The basic differential equation defines an infinite number of possible distorted maps or cartograms. The necessary next step is to add further conditions, and the one, which comes to mind immediately, is rather obvious. The distorted map should look as much like the usual map as possible or — in keeping with the present context — should maintain compactness as nearly as possible. These two statements turn out to be mathematically equivalent and require that the transformation be as nearly conformal as possible. The entire problem is thus given as

$$\text{Minimize} \iint_R [\partial u^2/\partial x + \partial u^2/\partial y + \partial v^2/\partial x + \partial v^2/\partial y]\, dx\, dy$$

$$\text{subject to}\ (\partial u/\partial x)(\partial v/\partial y) - (\partial u/\partial y)(\partial v/\partial x) - h(x, y) = 0$$

which requires only minor modification to be adapted to the terrestrial sphere or ellipsoid.

An analytical solution to this pair of equations is not known, but an iterative finite difference method has been programmed. The basic algorithm is as follows. Consider a rectangle on the map with bounding vertices u_i, v_i, i = 1, 2, 3, 4. The area of this rectangle is easily computed as a function of the values of vertex coordinates and can be compared with the desired area as defined by h, suitably normalized. By adding small increments Δu_i, Δv_i, i = 1, 2, 3, 4, to each of the vertex coordinates we can change the area of the rectangle. It is only necessary to compute the appropriate values for Δu_i, Δv_i, i = 1, 2, 3, 4. These values are obtained by writing the equation for the difference in area as a function of the modifications to the vertices. But there are eight values to be found, and only one equation. We now invoke the near conformality requirement which says that the change is to be a similitude. The single increment to the vertex coordinates is now found to be given by a quadratic function involving the difference between the desired and actual area of the rectangle.

This completes the adjustment for one rectangle. An interior vertex point, however, will be connected to four rectangles, and the latter may conflict in their desired adjustments. The vector mean of the displacements is used in this case, and the program cycles back for another iteration. The method is easily extended to cells of arbitrary shape. The program converges but it is slow, because before any vertex is adjusted a topological test must be performed to keep vertices from crossing the boundary of any cell, and boundaries are not allowed to cross over vertices. A mean square error criterion has been adopted to measure the efficacy of the iterations.

The United States has been used as a first example, beginning with an estimate of the number of people contained within each one degree quadrilateral of latitude and longitude. The computations start from an initial configuration in which the cell vertices are defined by integral values of latitude (φ) and longitude (λ), with a small adjustment for the mean latitude. The iterations produce a table of u, v coordinates for the bounding vertices of each cell; i.e., we have a one-to-one mapping $(\varphi, \lambda) \rightarrow (u, v)$. Using double bivariate interpolation, a map projection plotting program, and a magnetic tape containing an outline of the United States, a complete cartogram

can be plotted (see Figure 1). On this map we define a set of compact cells, hexagons, in u, v coordinates. Another program computes the inversion (u, v) → (φ, λ) and this allows us to map the hexagons back onto the regular United States map. Each cell now contains an equal number of people, and this result is exact to the extent that the iterations have converged. It is obviously not valid in the oceanic areas, where some population has been allocated for aesthetic reasons.

The southern portion of the state of Michigan has been used for a second example. The requirement was to obtain nineteen political districts of equal population, without splitting any of the counties. Three counties already contained more than one-nineteenth of the population each, so that the exact solution space was a null set, which is not of great concern in illustrating the method. The diagram shown is the result of only eight iterations, and the county sizes are thus not yet exactly proportional to their populations (see Figure 2). This result could be improved by further iterations. Now any proposed plan of districts might be evaluated visually by the drawing of boundaries on this map. Alternately, the manual construction of districting plans might be undertaken directly on such maps, with the results later transferred to more conventional maps. Figure 3 illustrates this process, which of course is exactly equivalent to the inversion procedure demonstrated above for the United States using hexagons. First, a square grid is placed over the cartogram. Since the required number of districts (19) is prime, a grid of twenty cells has been used. If we could do away with the county boundaries the regular lattice would in fact provide a serviceable set of districts. Since we cannot, by the assumptions of the problem, get rid of the county boundaries, let us consider the grid as suggestive of what the final appearance of the districts should be. More precisely, let us attempt to find a set of districts the boundaries of which depart as little as possible from the grid. "As little as possible" can be taken in the least squares sense, meaning that some measure of mismatch, e.g. the area between the boundaries of the districts and the edges of the cells, must be minimized. A formal solution of the problem stated in this manner does not seem easier than the initial problem. Visually and heuristically we begin by attempting to match the edge of the grid with the edge of the map. (The map can be made to fit exactly into a rectangle of the proper proportions, but this has not been done in the example.) Now pick those county lines that fall near grid lines, and do a lot of fudging to obtain the requisite number of districts (19 in the present instance — recalling that one of the counties is already bigger than three optimal districts and that we do not allow splitting of counties, by our assumptions). Finally transfer the lines back to the original map. The mean percentage error of the districting on this first attempt is 44.6%. This result is atrocious, even though most of the error must, at least in this instance, be attributed to the initial conditions of the problem.

As a third example, assume that the population density of a city can be described to an arbitrary degree of accuracy by a mathematical equation, and that the first term in such an expansion (see Tobler 1969) is $D = f(r) = 6P(r - R)^2/\pi R^4$, where

$\qquad$ D = number of persons per square kilometer

P = total population

r = distance in kilometers from the center of town

R = 0.035P$^{0.44}$, the radius of the town

π = 3.14159.

In this "parabolic" town, densities do not vary with direction, so that a natural, not necessarily optimal, solution to the defining differential equation in polar coordinates (r', θ') is given by

$$r' = [P/\pi\, (3r^4/R^4 - 8r^3/R^3 + 6r^2/R^2\,)]^{1/2}\,,$$

$$\theta' = \theta.$$

Figure 4 shows this transformation applied to a simple street map of the city of Ann Arbor, whose population is 100,000. This population cartogram may be used as were the previous maps if one keeps in mind the several approximations that were invoked during its construction. The polar coordinate form of solution is especially advantageous when pie-shaped districts are specified by the legal system.

The foregoing three examples demonstrate application of a continuous transformation to a districting problem with data given for a cellular lattice, for irregular polygons, and by an approximating equation, at national, regional, and local scales. A definite improvement in accuracy and especially in computational speed would be achieved if an analytical solution were available for the system of differential equations.

[1] *Annals*, New York Academy of Sciences, Vol 219 (1973), pp. 215-220.

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FIGURE 1. The left column pertains to the usual type of map; the right column, to the population cartogram. The cartogram has converged to 68% of its desired accuracy after 99 iterations. Top row: one degree latitude and longitude graticule for the secant plate carée map projection and for the cartogram, respectively. Middle row: maps corresponding to the above. Bottom row: compact regions of equal population for each of the two maps.
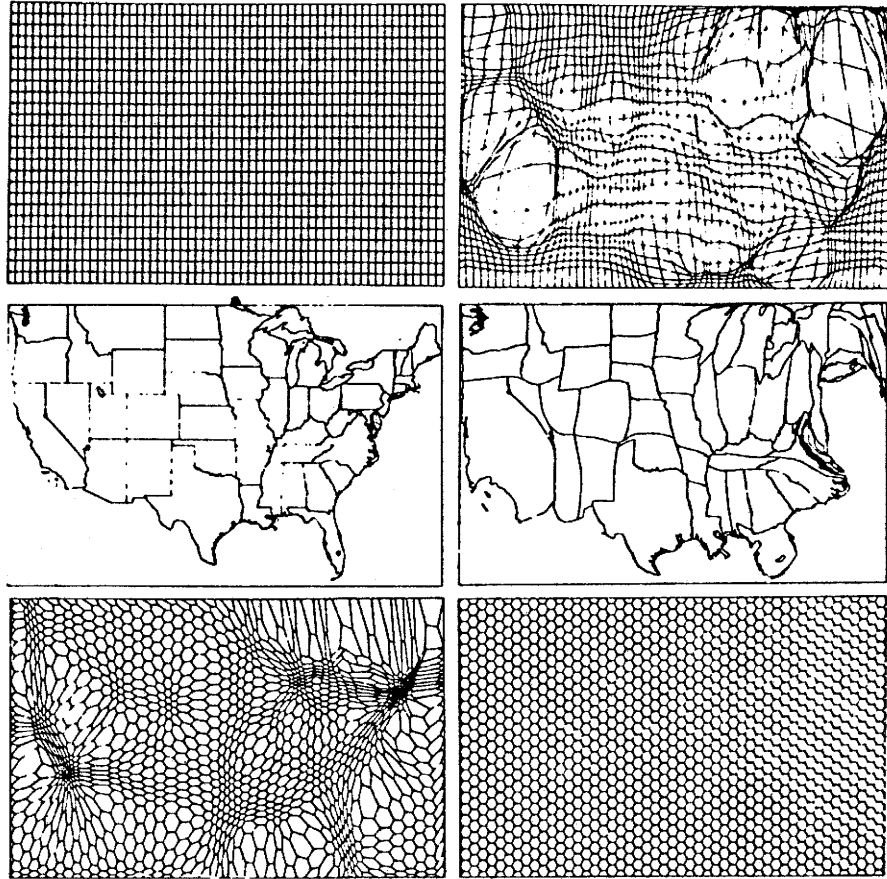
FIGURE 2. Southern Michigan: normal county map (left) and 1970 population cartogram after eight iterations.
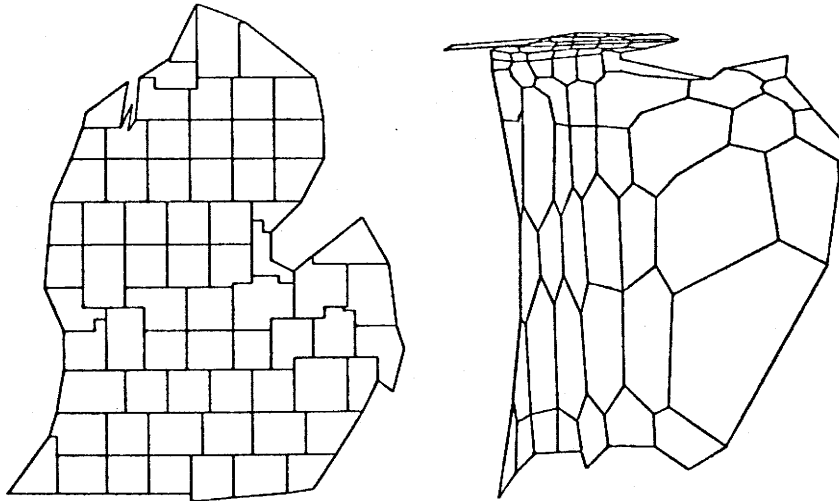

FIGURE 3. Districting on a cartogram. From left to right: approximation by a regular grid, accommodation to the required boundaries, and transfer to the original map. {The halftone lines on the maps did not reproduce well}
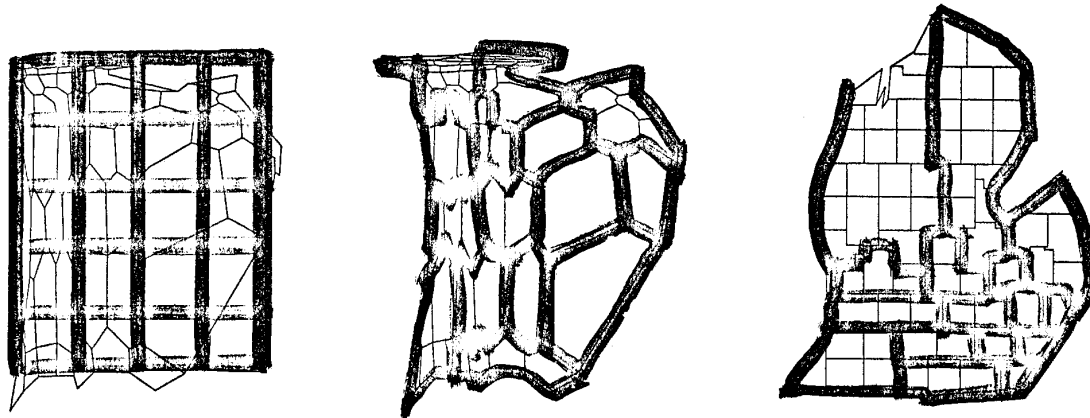
FIGURE 4. Left: conventional map of Ann Arbor. Right: population cartogram of Ann Arbor.