The Care and Feeding of Vector Fields

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A Better Title Might Be

Creating, using, manipulating, and inverting vector fields.

Abstract

Objects and observations used in GIS are most often categorical or numerical. An object less frequently represented has both a value and a direction. A common such object, familiar to all, is the slope of a topographic surface. However numerous additional instances give rise to vectors. Well-known operations, such as filtering and interpolation, can be applied to vectors. There are also analyses unique to vectors and vector fields. Some of these result in a further generalization, objects that have different magnitudes in all directions, a.k.a. Tensors.

Subjects To Be Covered
Partial List

Conventional sources of vector fields

What can be done with vector fields

Increasingly abstract examples

Calculating potential fields from tables

Resolution and its effects

In GIS It Is Common to Refer to Rasters and Vectors.

These refer to the format of the data

This is NOT what my talk is about!

Rather I am looking at the sequence

Scalar - Vector - Tensor

The Most Frequent Data in a GIS Are

Categorical data

or

Scalar data

Examples of Categorical Data Are:

Nominal classes such as land use or soil type.

These can be given as classes within polygons or by 'pixels' in a raster.

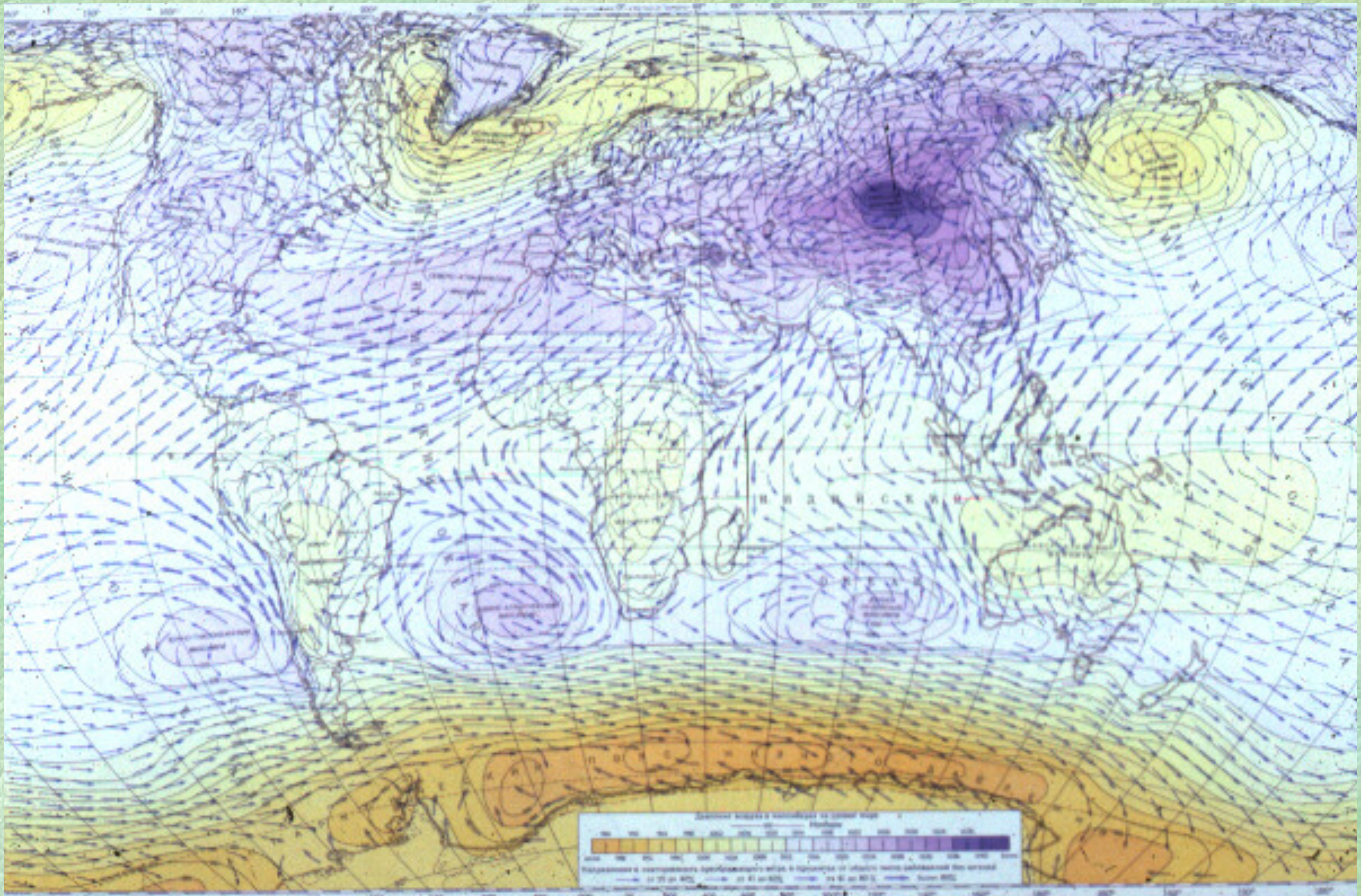
Single Numbers at Every Place Are Examples of Scalar Data.

As for a raster (or TIN) of topographic elevations, or population defined for polygons, etc.

Two Numbers at Every Place Are Examples of Vector Data.

Wind speed and direction is a good and well known example of a vector field.

World Wind Pattern



“Field” Refers to the Notion That the Phenomena Exist Everywhere.

Thus we can have:

Categorical fields - soil type

Scalar fields - topography

Vector fields - wind, currents

Tensor fields - terrain trafficability

It Is Not Implied That the Values Have Been Measured Everywhere

But that they can conceptually exist everywhere.

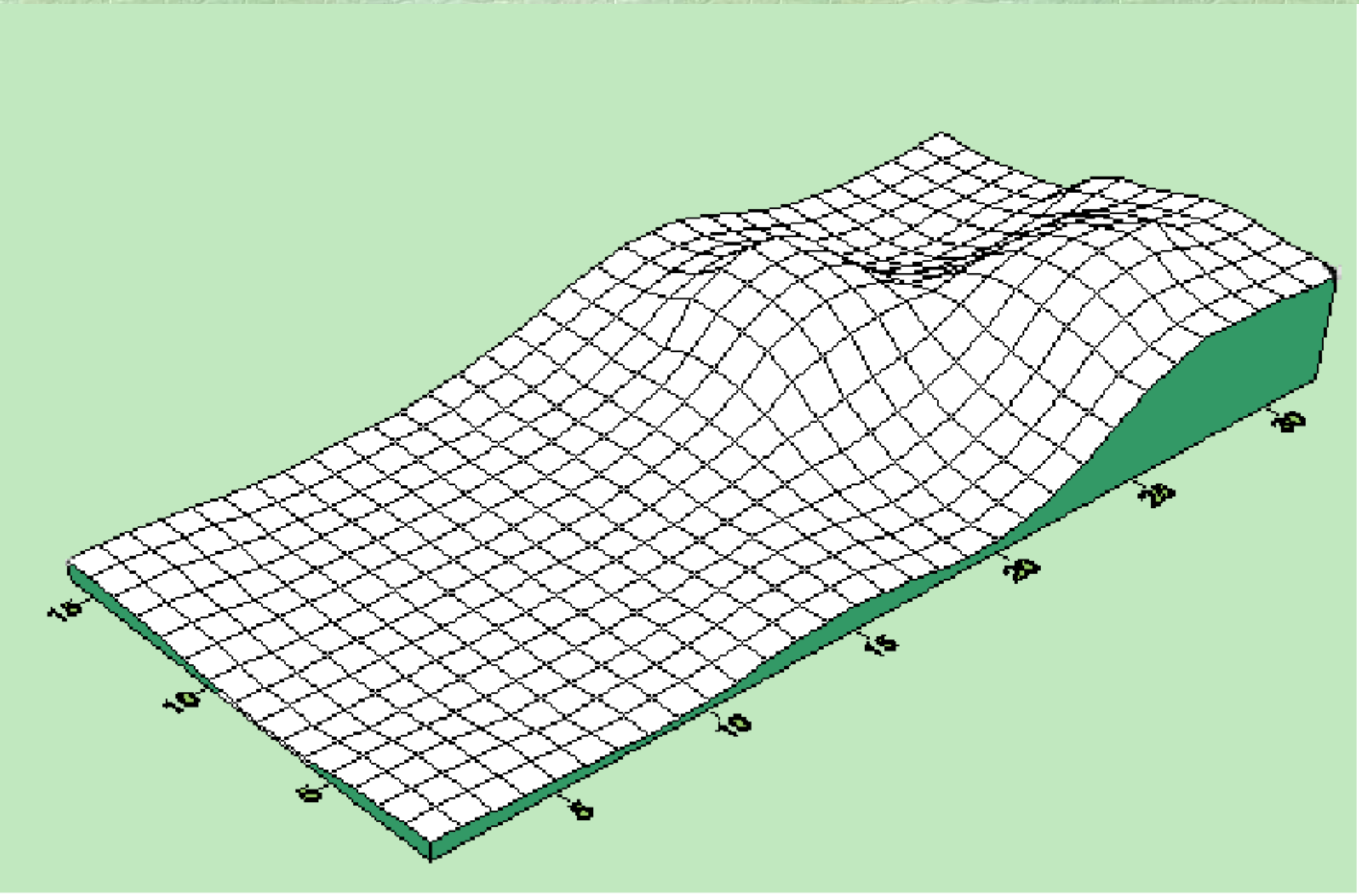
So a vector field might be sampled, and known, only at isolated locations, or at the vertices of a regular lattice or other tesselation.

A Familiar Vector Field Can Be Defined For Topography

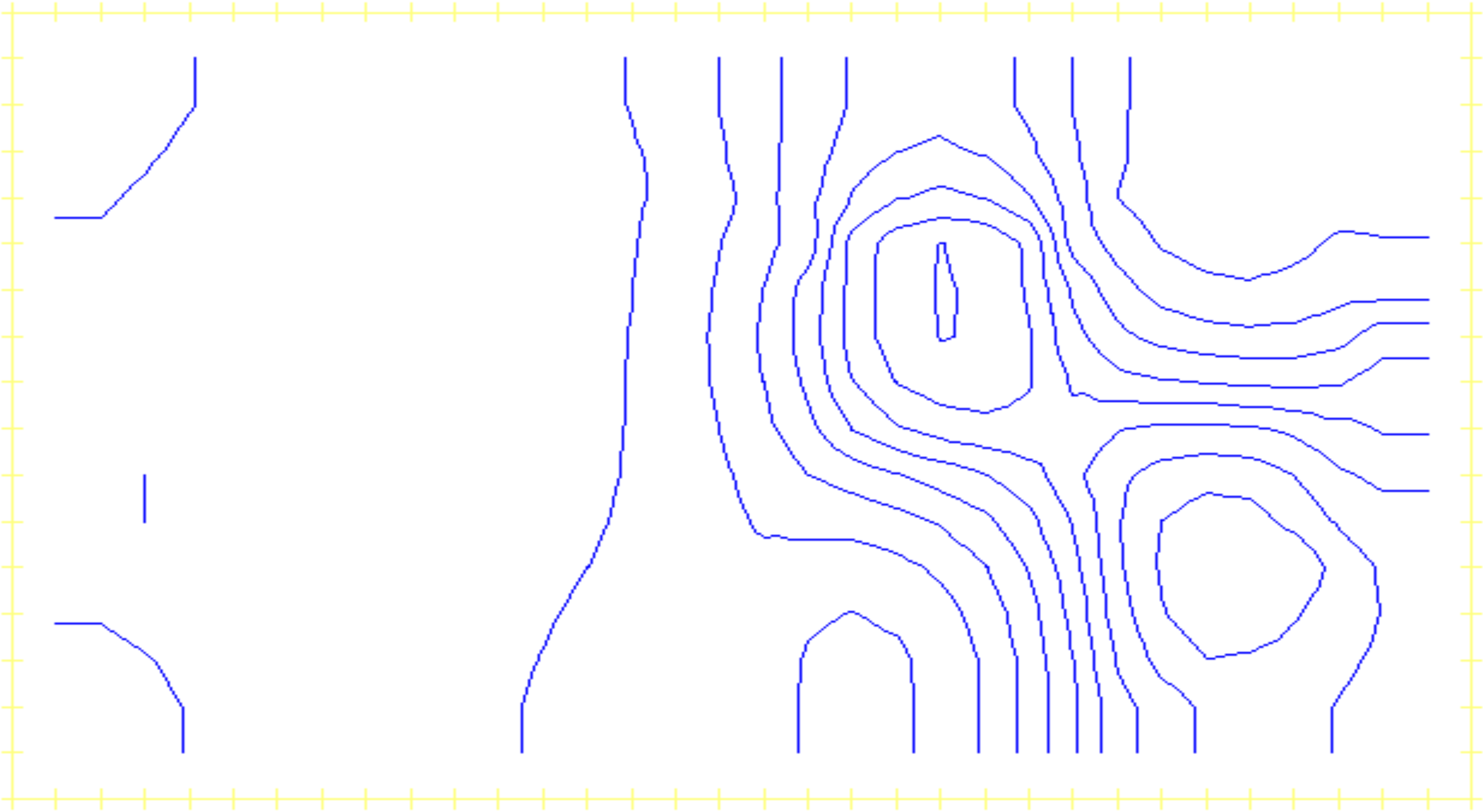
The slope of a topographic surface gives rise to a vector field.

For example if we start with

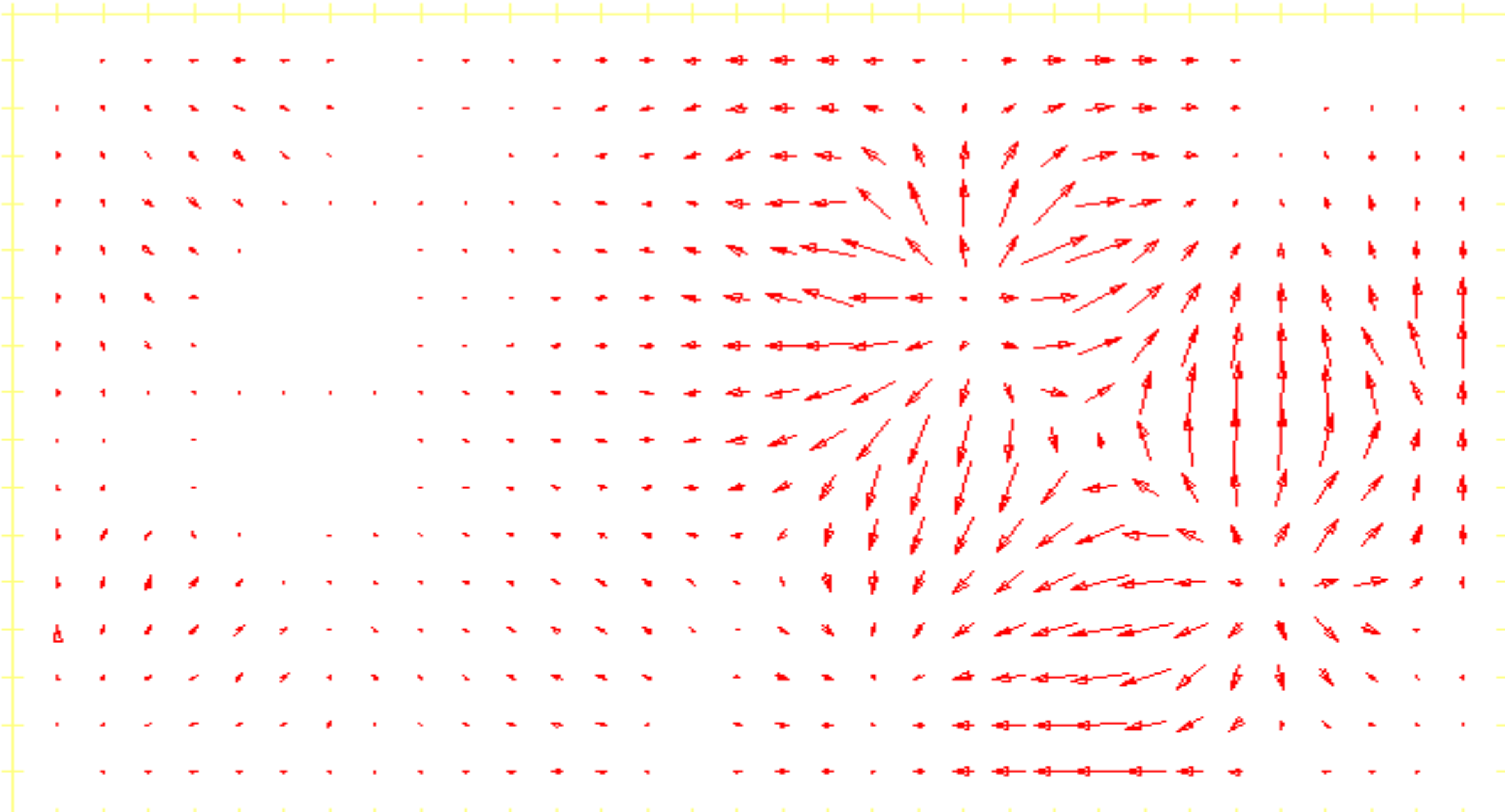
A Simple Topographic Surface



Here It Is Shown By Contours

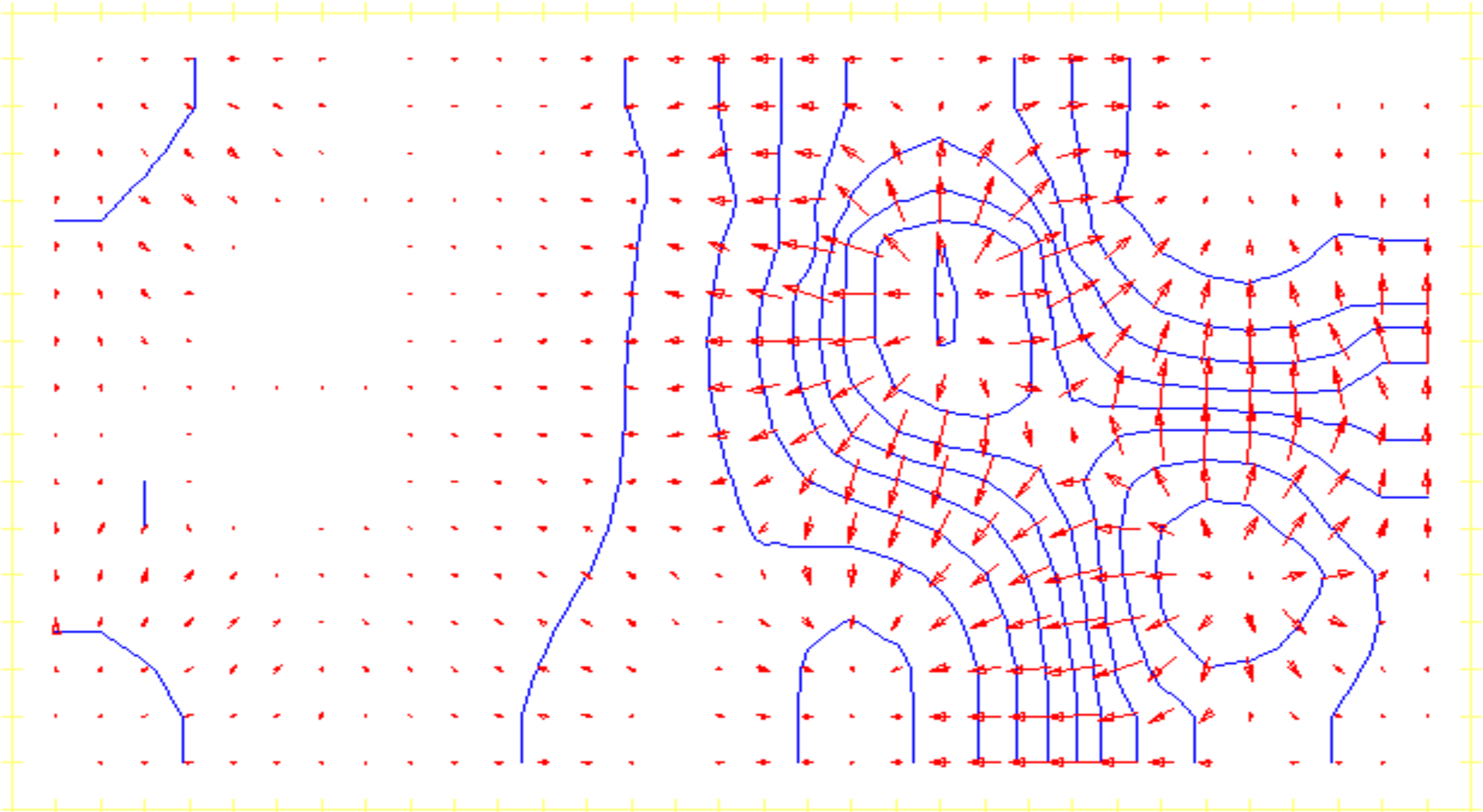


And Here Are The Gradients: A Field Of Vectors



Here Are Both Contours And Gradients

The gradients are orthogonal to the contours



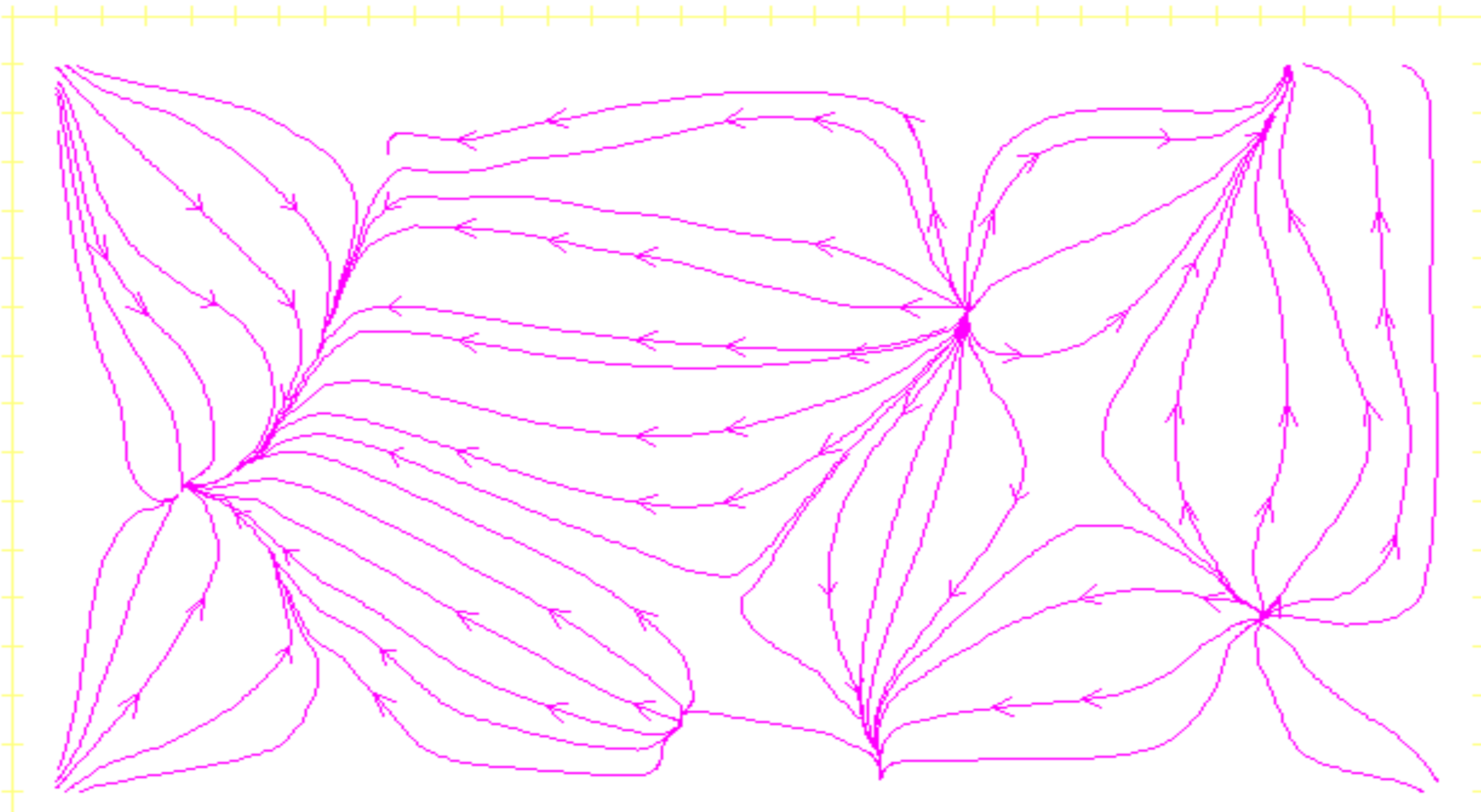
The Gradient Field Has the First Partial Derivatives of the Topography As Its Components.

The derivatives of the vector field give rise to further objects.

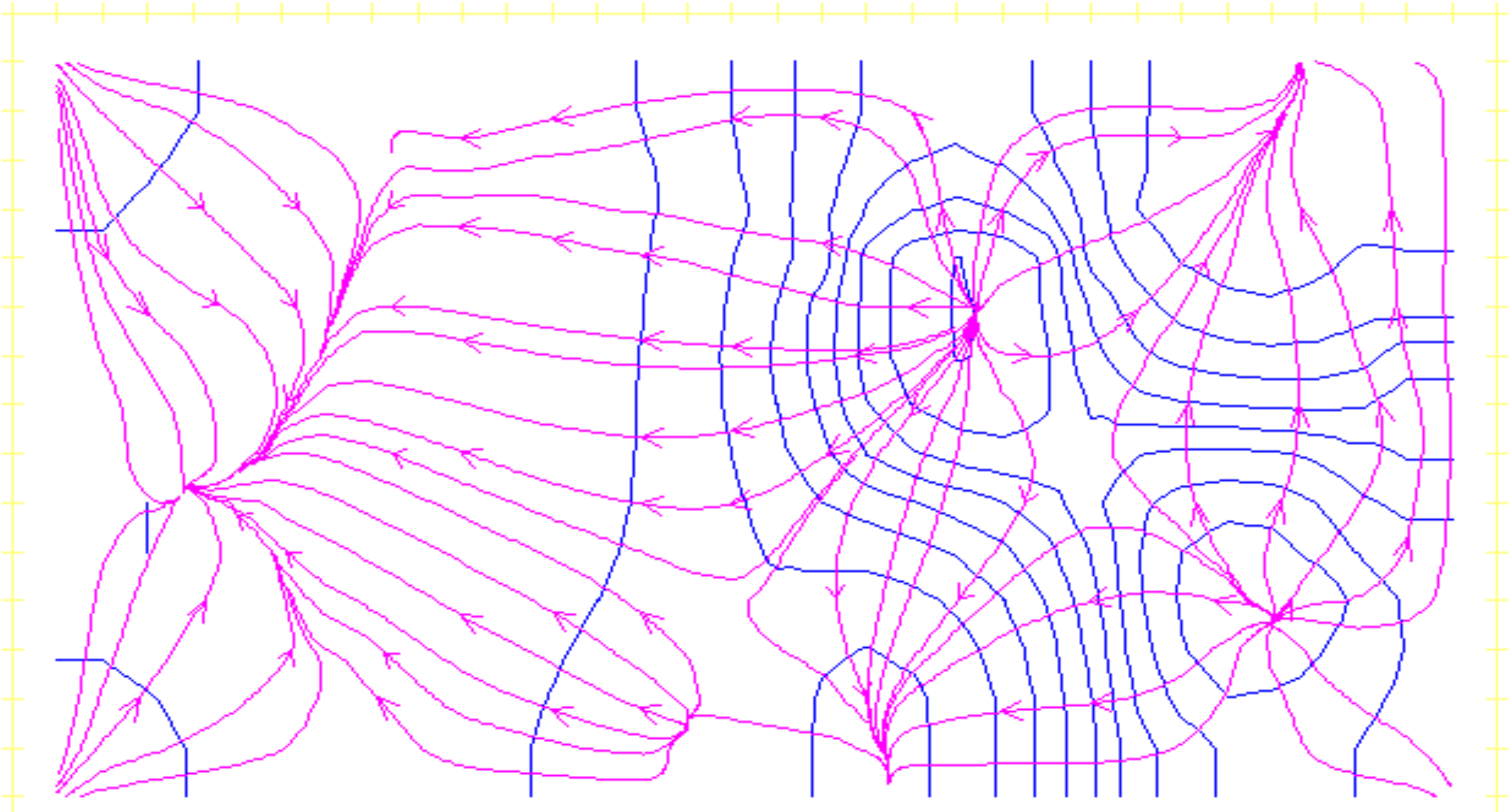
For example, second derivatives are often used in geophysics to determine the spatial loci of change. They are similar to the Laplacian filters used in remote sensing applications.

There may be further uses of these higher derivatives.

From Vector Field to Streaklines



Contours and Streaklines



The Streaklines Are Constructed Using the Gradient Vectors

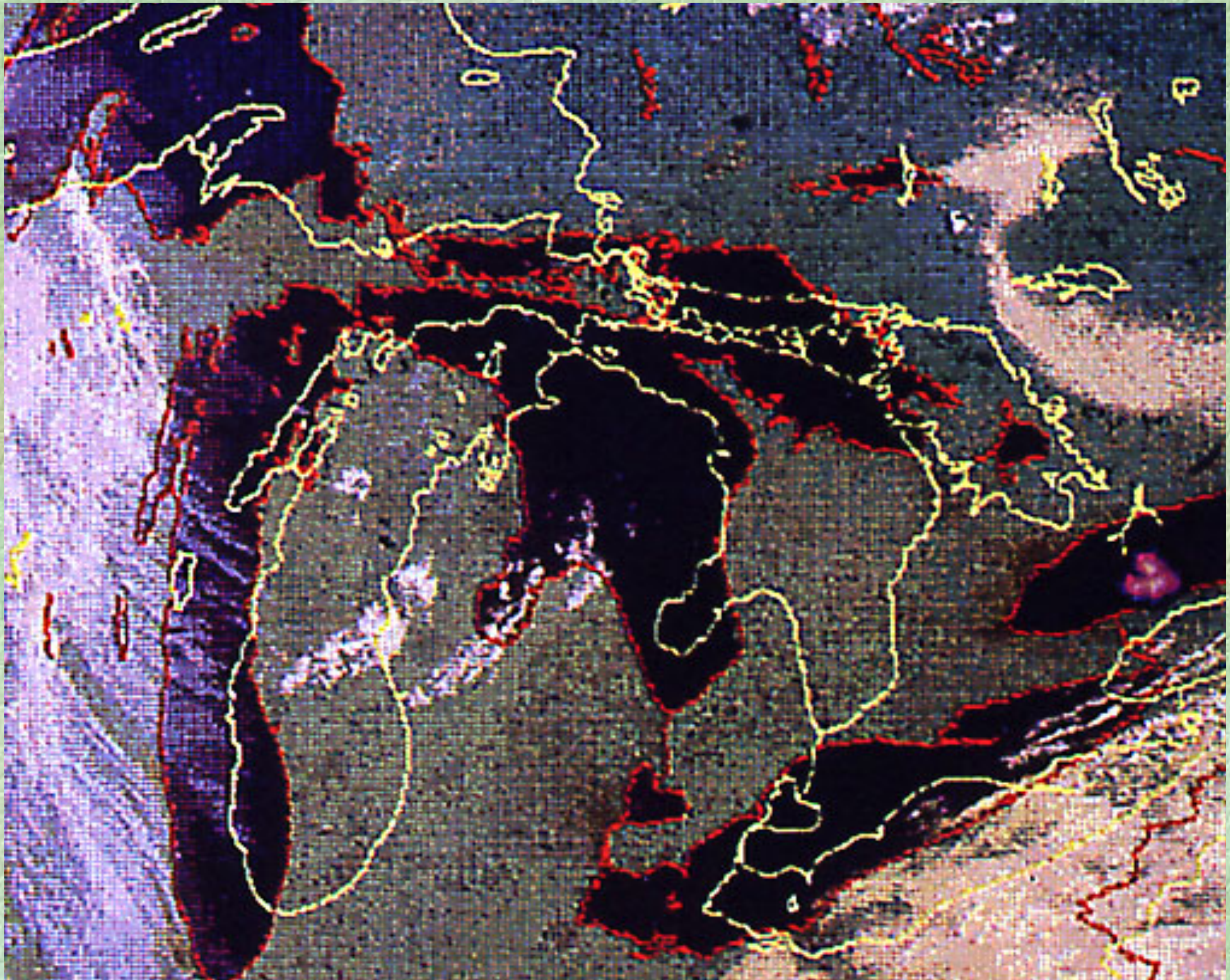
As such they are also orthogonal to the contours.

Basins may now be delineated

Those of you working in physical geography will recognize that producing stream traces is a little more complicated than this. There is a large literature.

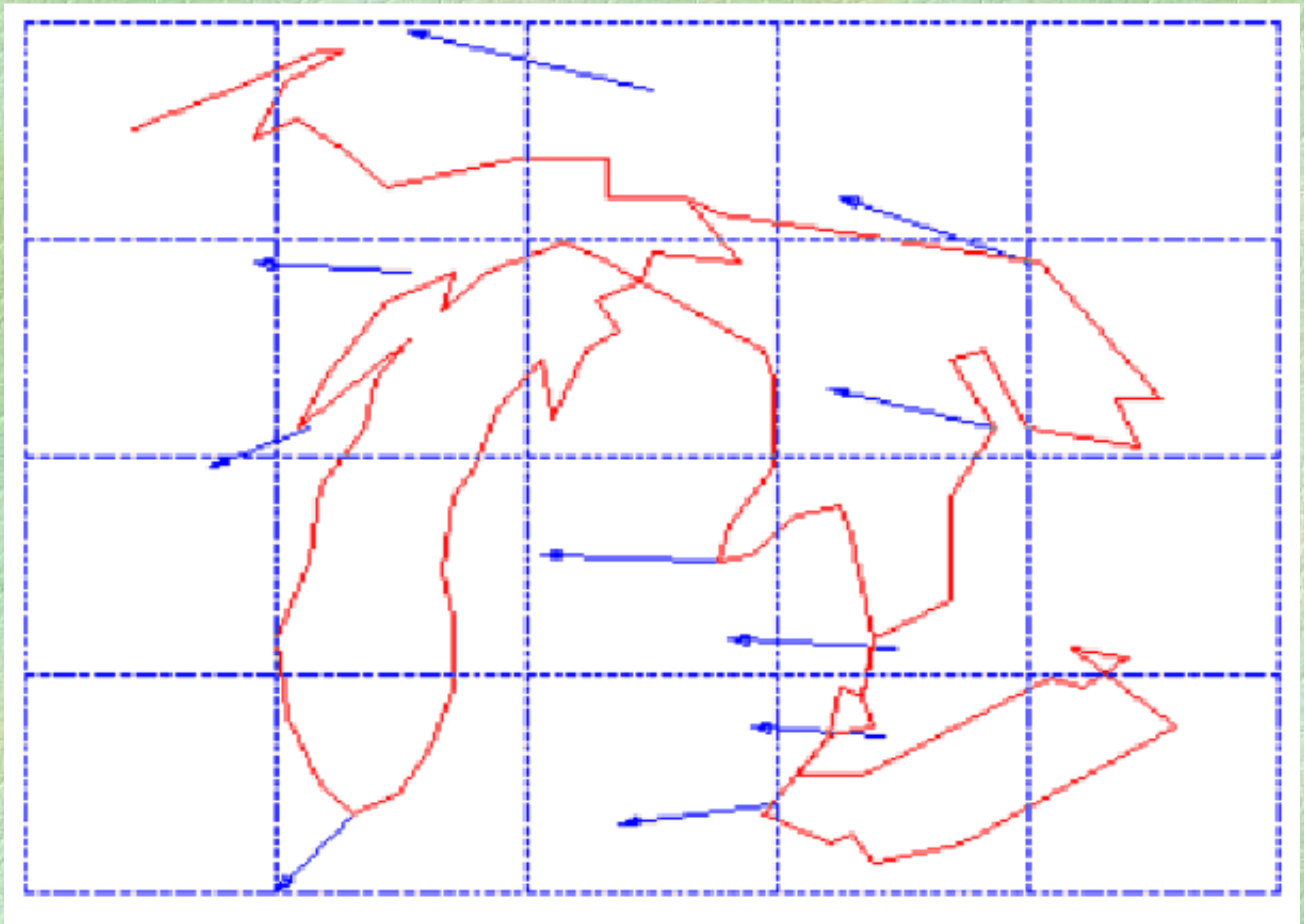
Vectors Also Appear in Map Matching.

Here is an example: Map and Image



The Difference Between The Map and the Image

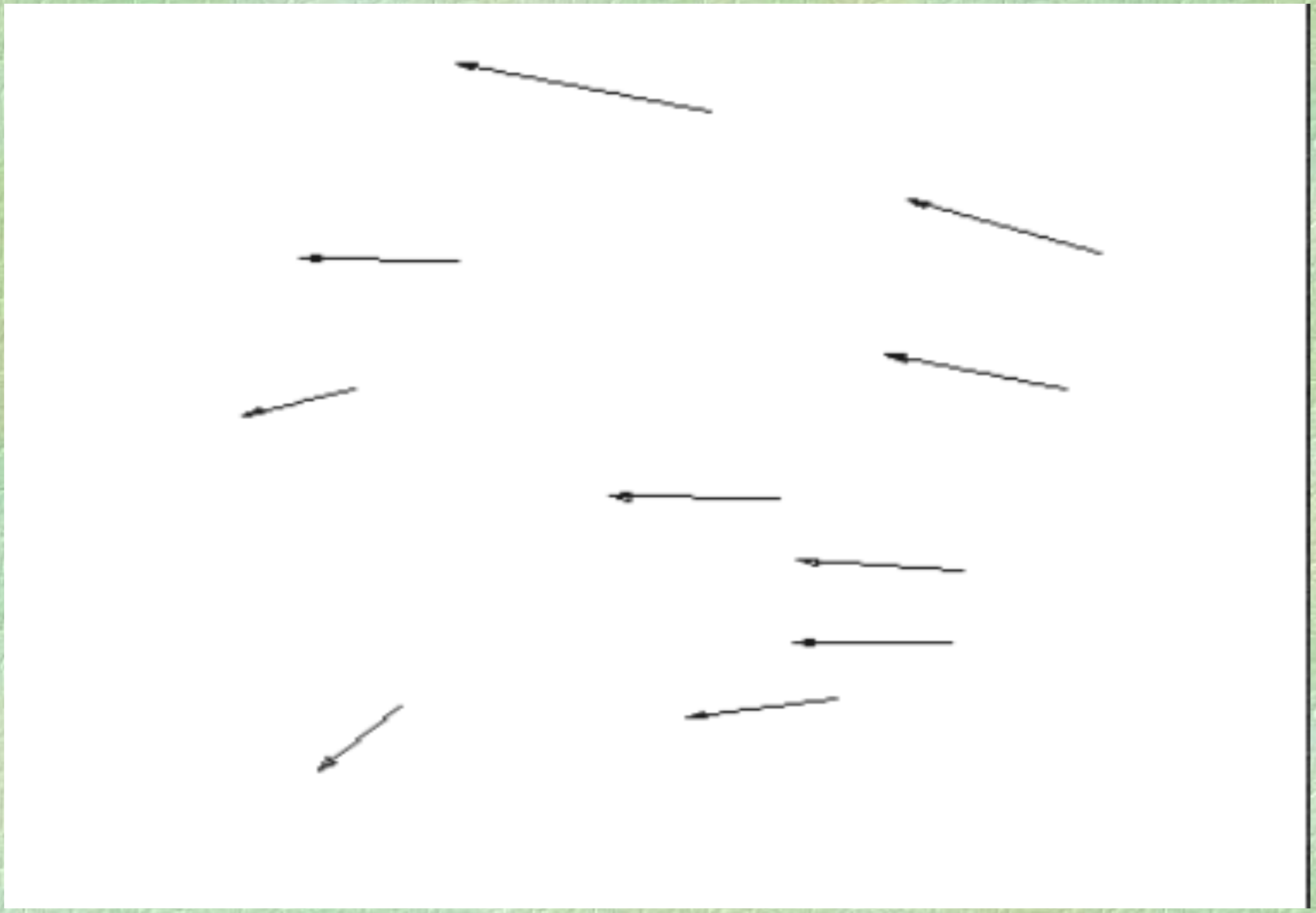
Shown as discrete vectors



The Vector Field Given as Map to Image Displacements

Coordinates	
Map	image
25 11	18 03
74 28	59 29
21 51	12 47
52 86	30 92
63 12	49 10
58 37	42 38

Difference Vectors
by themselves, without the grid



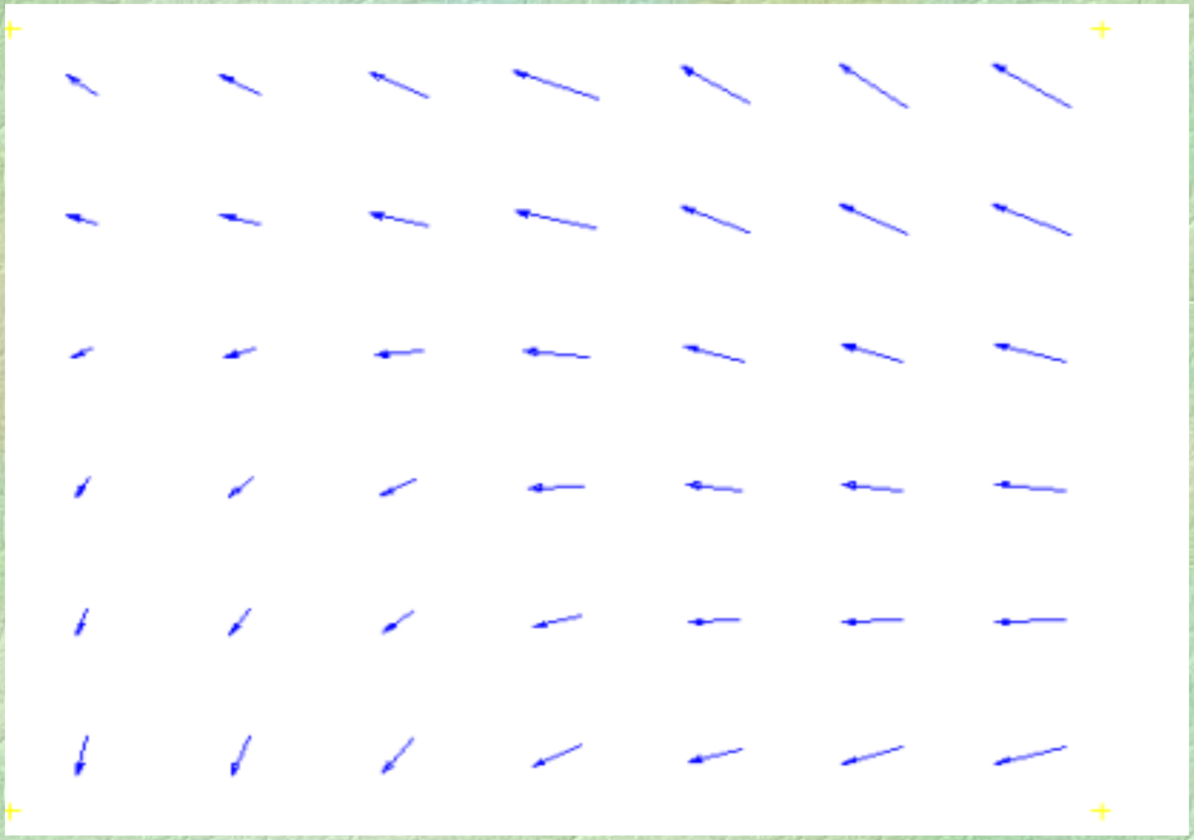
Scattered Vectors Can Be Interpolated to Yield a Vector Field

Inverse distance, krieging, splining, or other forms of interpolation may be used.

Smoothing or filtering of the scattered vectors or of the vector field can also easily be applied. This is done by applying the operator to the individual vector components.

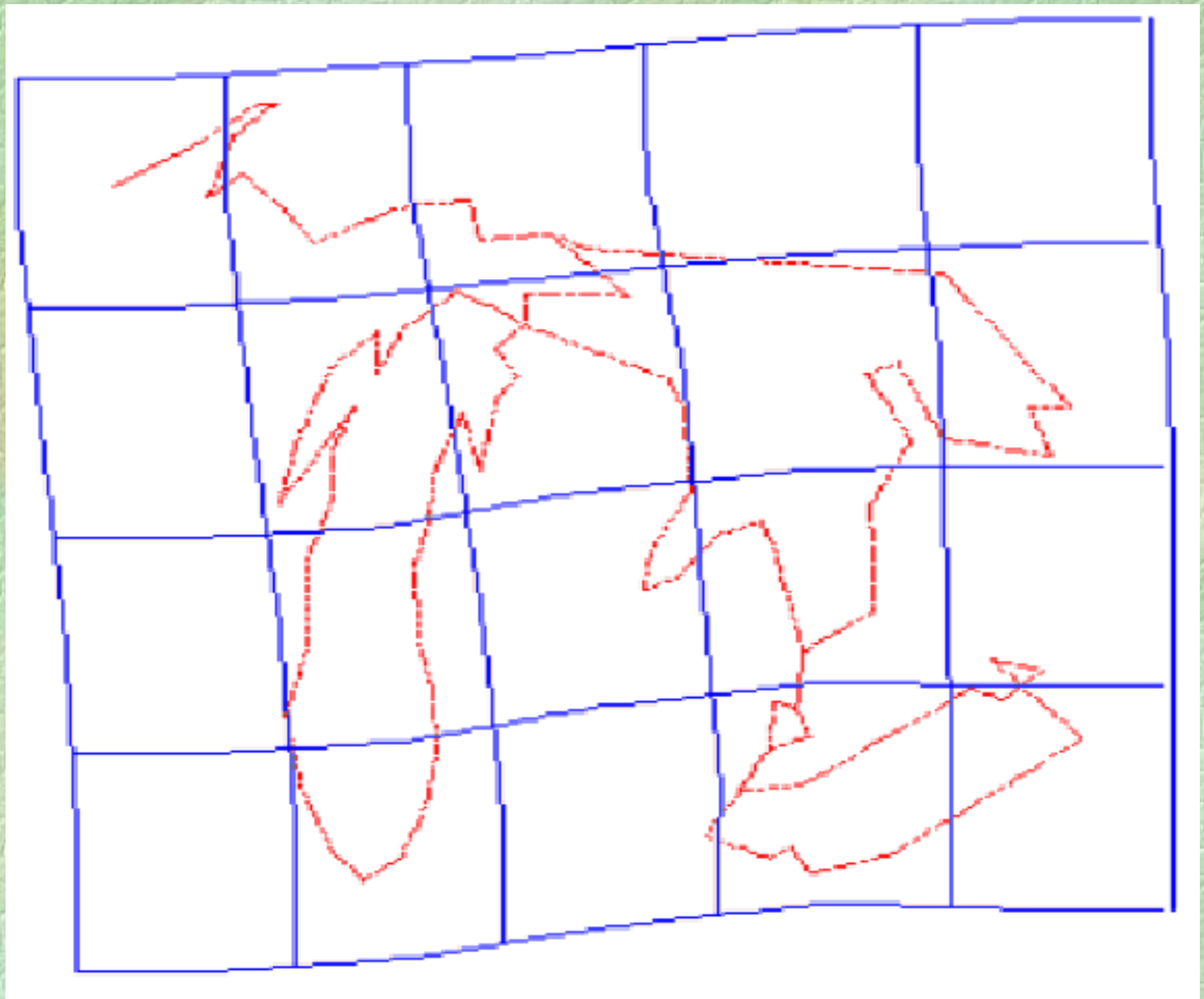
Or treat the vectors as complex numbers with the common properties of numbers.

Interpolated Vector Field



Great Lakes Displaced

The grid has been ‘pushed’ by the interpolated vector field



Here Is an Example From the Field Known As ‘Mental Mapping’

A list of the sixty largest US cities, in alphabetical order, is given to students.

Cities and Locations

Coordinates not given to students.

1	AKRON	41.066	-81.516
2	ALBUQUERQUE	35.083	-106.633
3	ATLANTA	35.749	-84.383
4	AUSTIN	30.299	-97.783
5	BALTIMORE	39.299	-76.633
6	BIRMINGHAM	33.499	-86.916
7	BOSTON	42.333	-71.083
8	BUFFALO	42.866	-78.916
9	CHARLOTTE	35.049	-80.833
10	CHICAGO	41.833	-87.749
11	CINCINNATI	39.166	-84.500
12	CLEVELAND	41.499	-81.683

Instructions to the Students

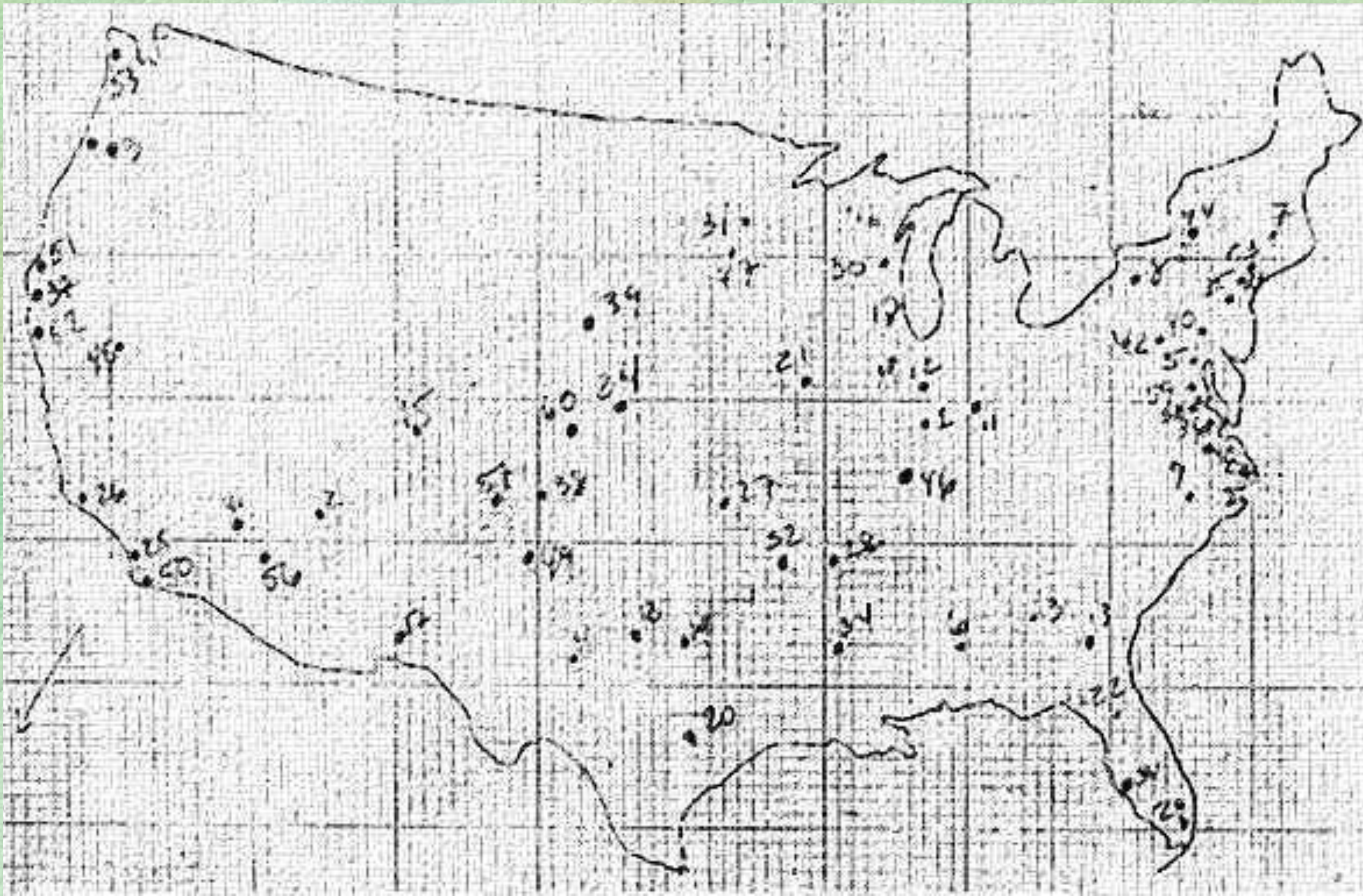
Work without any reference materials

Use Graph Paper, wide Margin at top.

Plot Cities with ID Number on the Graph Paper.

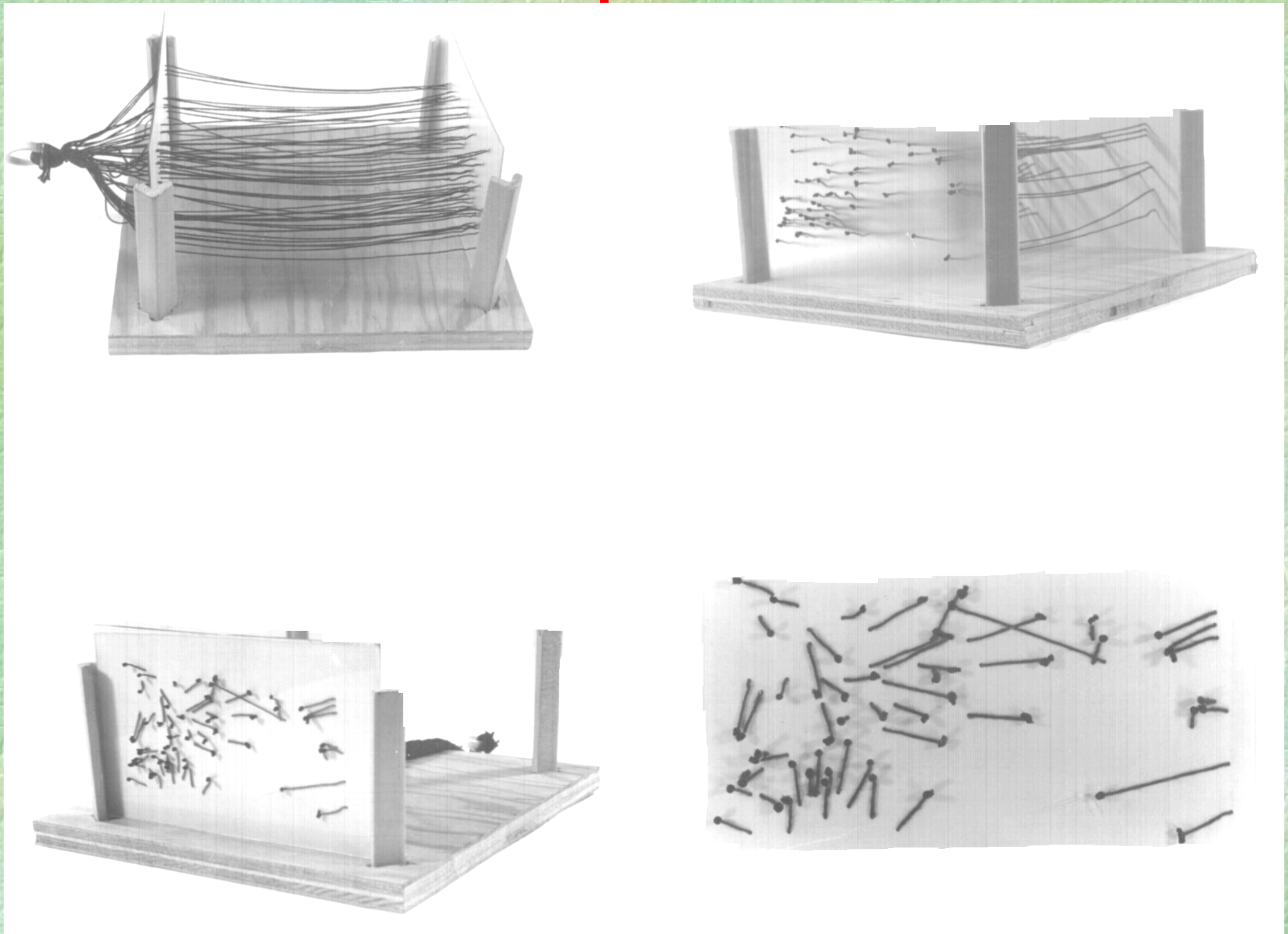
USA Outline may be drawn, but is not required.

An Anonymous Student's Map



To illustrate the scoring concept for students I have built

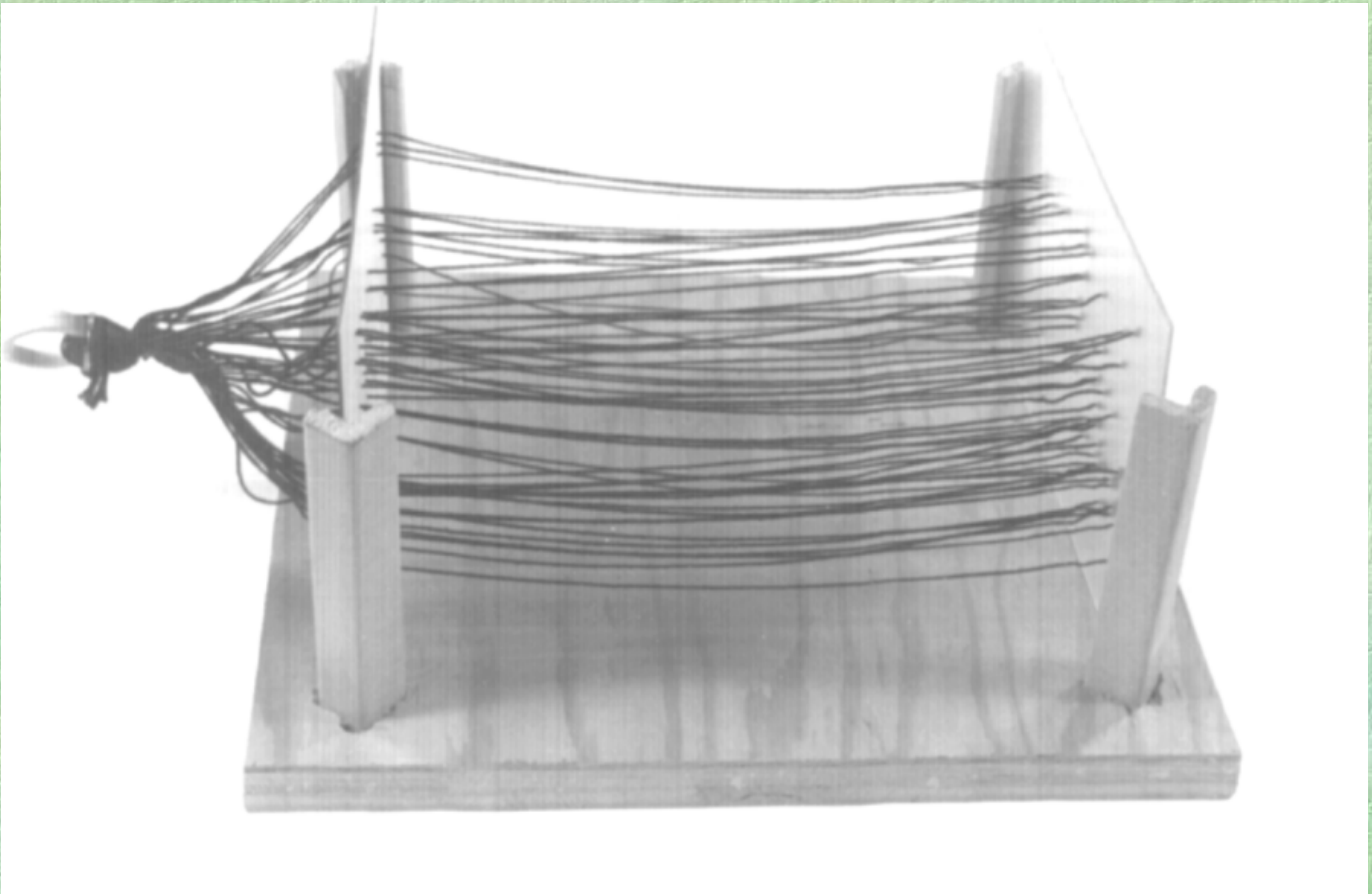
The Map Machine



The Map Machine

Detail View 1

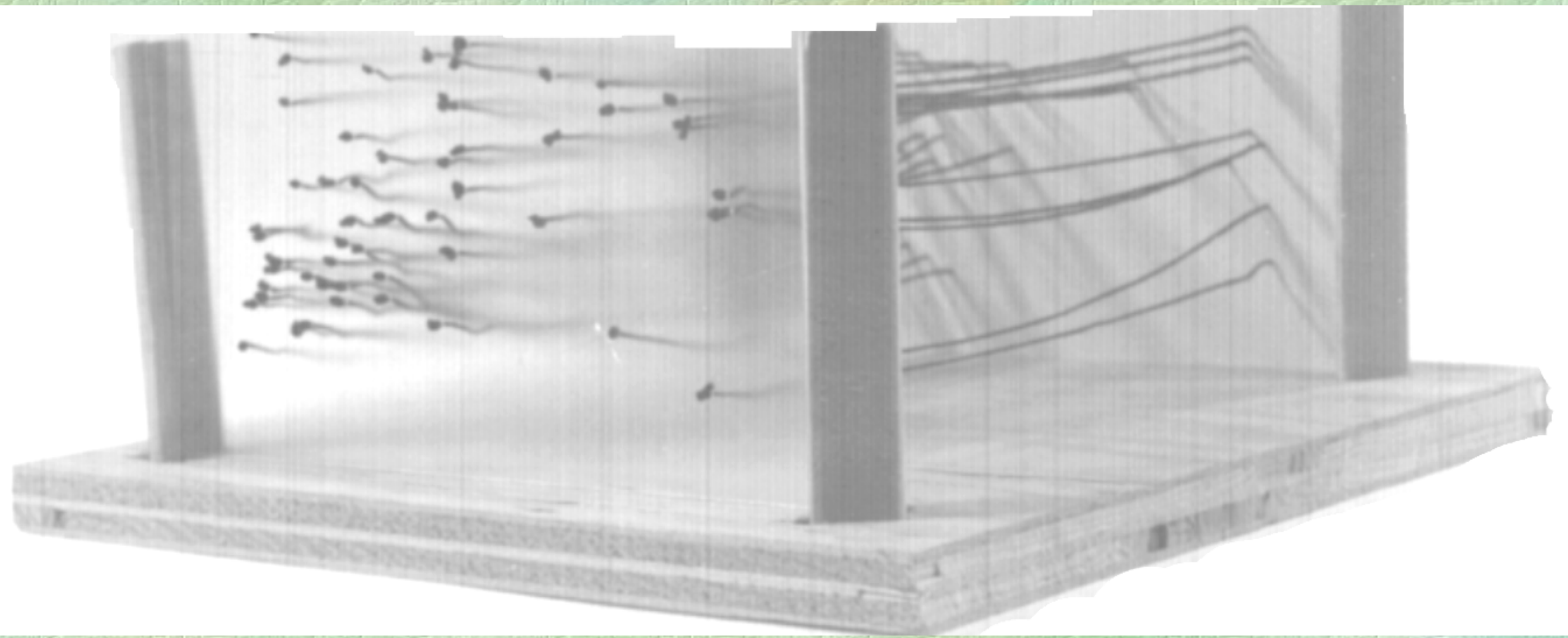
Showing the one to one correspondence between the images



The Map Machine

Detail View 2

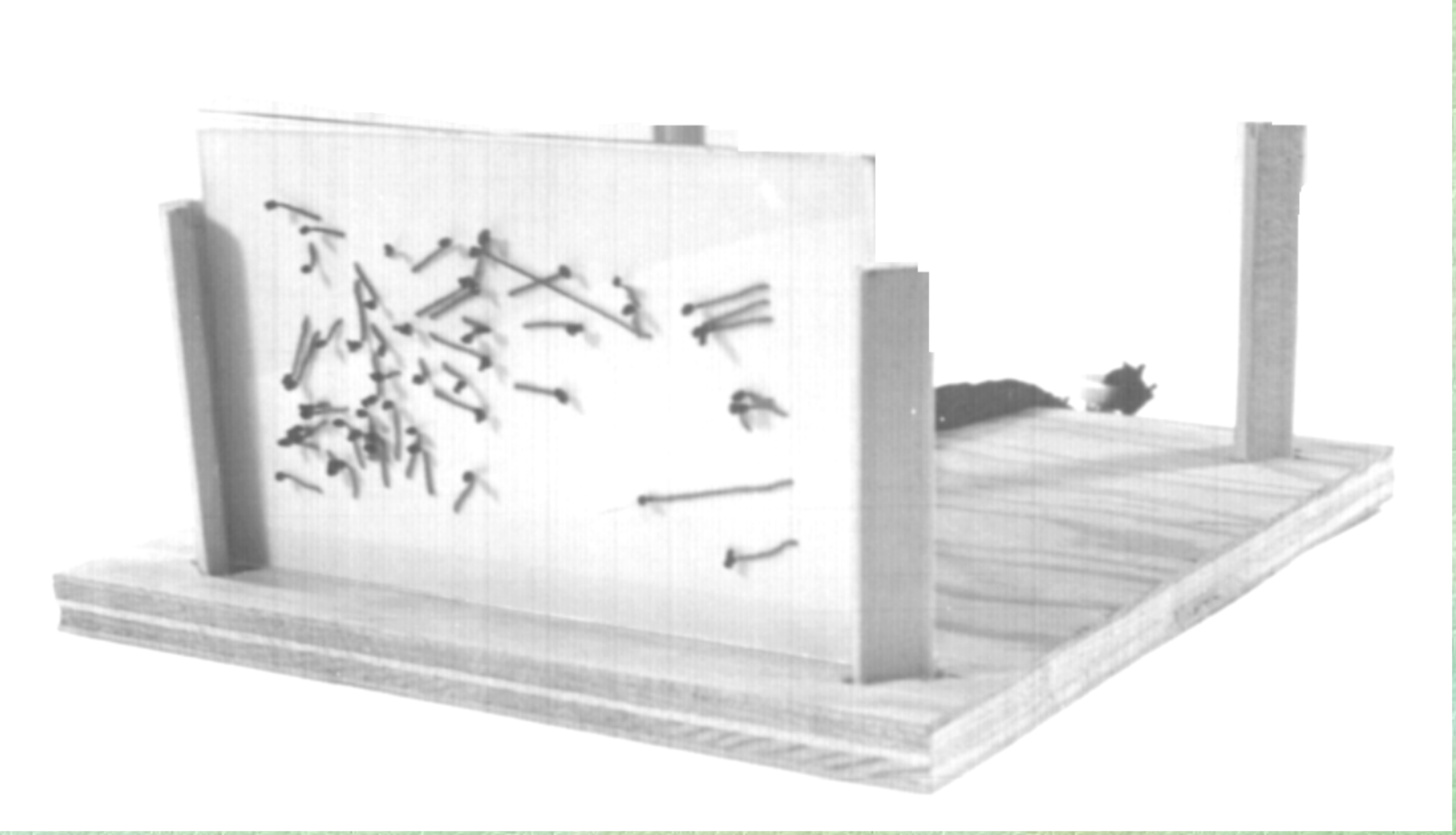
The front panel is transparent, back panel is white, strings are black



The Map Machine

Detail View 3

Releasing the back panel and pulling the strings together



The Map Machine

The Final View

corresponds to the computer image of displacements

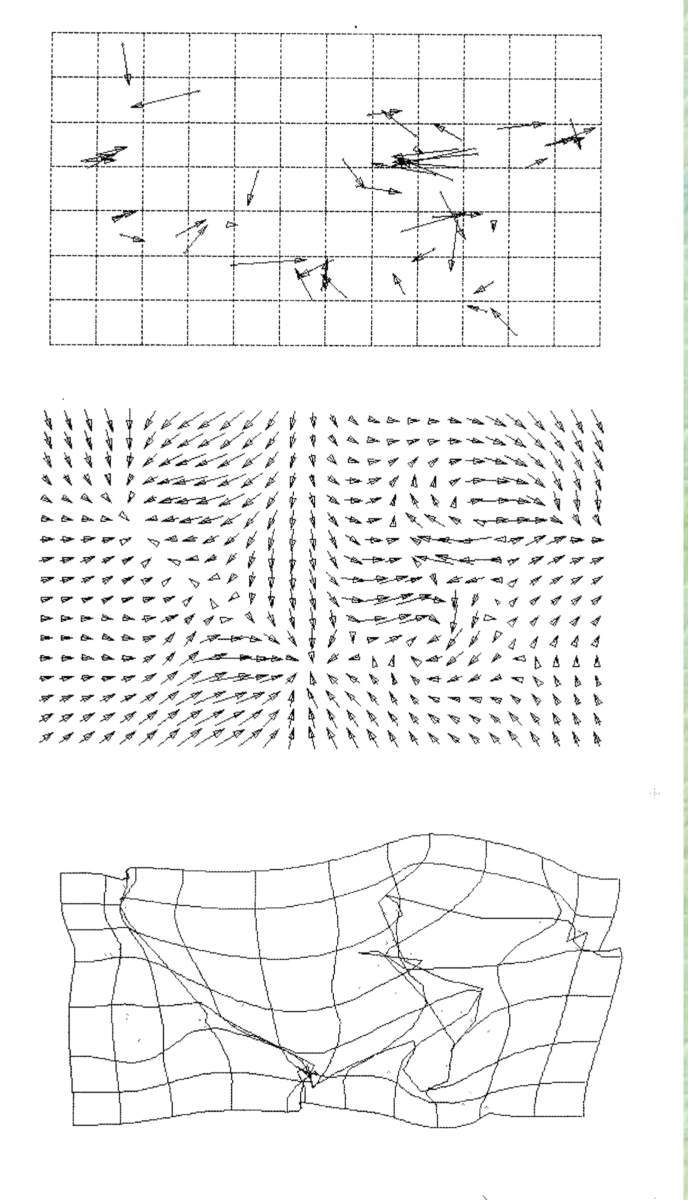
The Student Map Shows Displacement Vectors

These vectors could also show change of address coordinates, due to a move.

Or they could be home to shopping moves, etc.

Thus there are many possible interpretations of this kind of vector displacement

Analysis of Student Data



Displacement vectors

Interpolated vectors

Displaced grid

The displaced grid could be used to interpolate a warped map of the United States.

Given the severe displacements the map would need to overlap itself

With Student Maps In Hand

How to score?

Compute correlation, R^2, between actual and student estimates? How to do this?

Correlation between scores of different students? Factor analyze?

Compute vector field variance, etc., to determine degree of fuzziness?

Average vectors over all students?

It is often the case that one has several vector fields covering the same geographic area. A simple example would be wind vectors and ocean currents. How can these different fields be compared?

Is There a Method of Computing the Correlation Between Vector Fields?

The question comes up not only in meteorology and oceanography but also for the comparison of the student's maps, for comparison of old maps, and in many other situations. There are in fact such correlation methods, and associated with these are regression-like predictors. Statistical significance tests are also available.

B. Hanson, et al, 1992, "Vector Correlation", *Annals*, AAG, 82(1):103-116.

More Questions

What about auto-correlation within a vector field?

Or cross-correlation between vector fields?

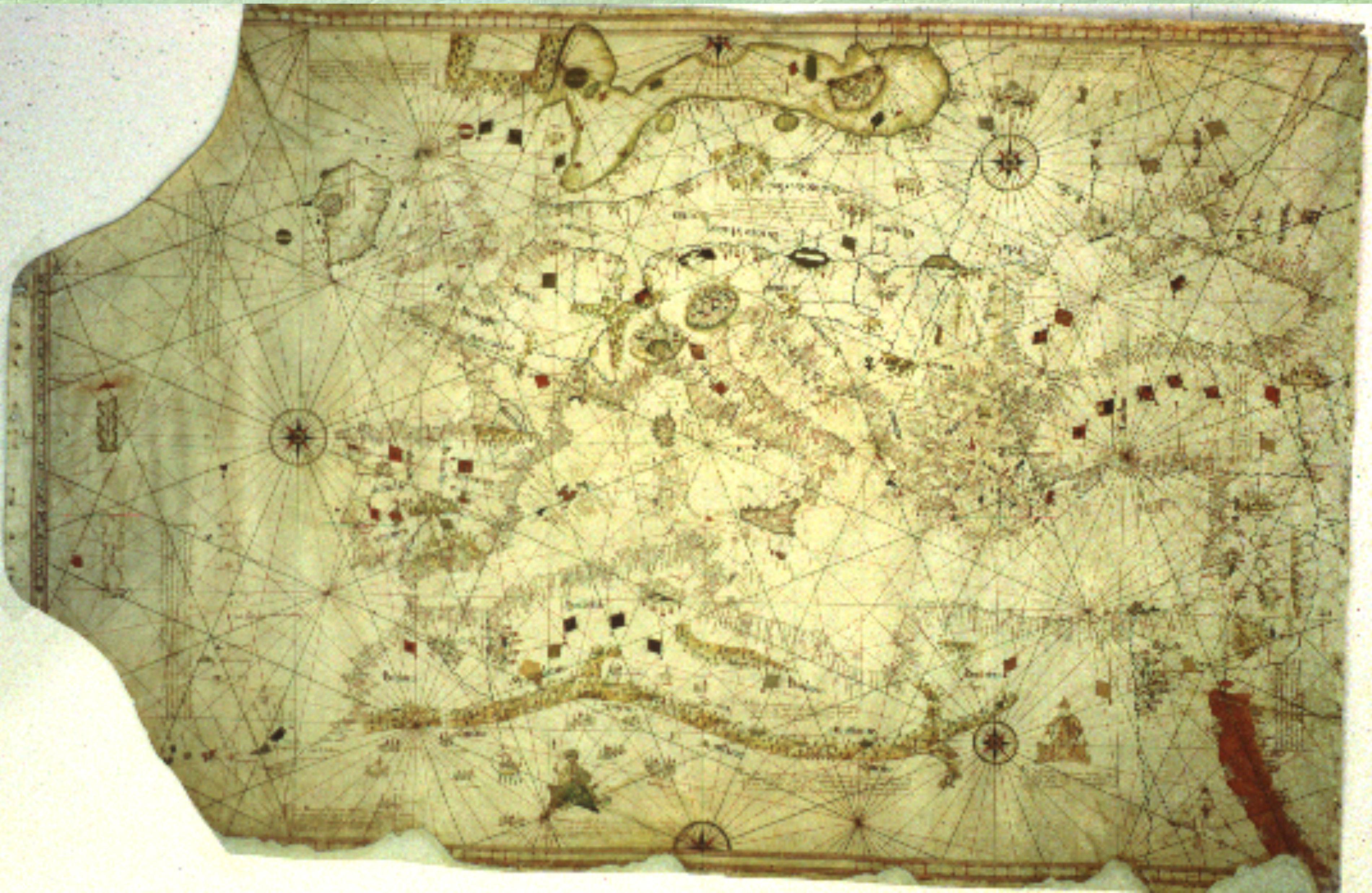
Or vector field time series?

But those are topics for another day.

I also have an interest in the structure of old maps.

Here is an analysis of one that is over 500 years old.

Benincasa Portolan Chart



Coordinates From Scott Loomer

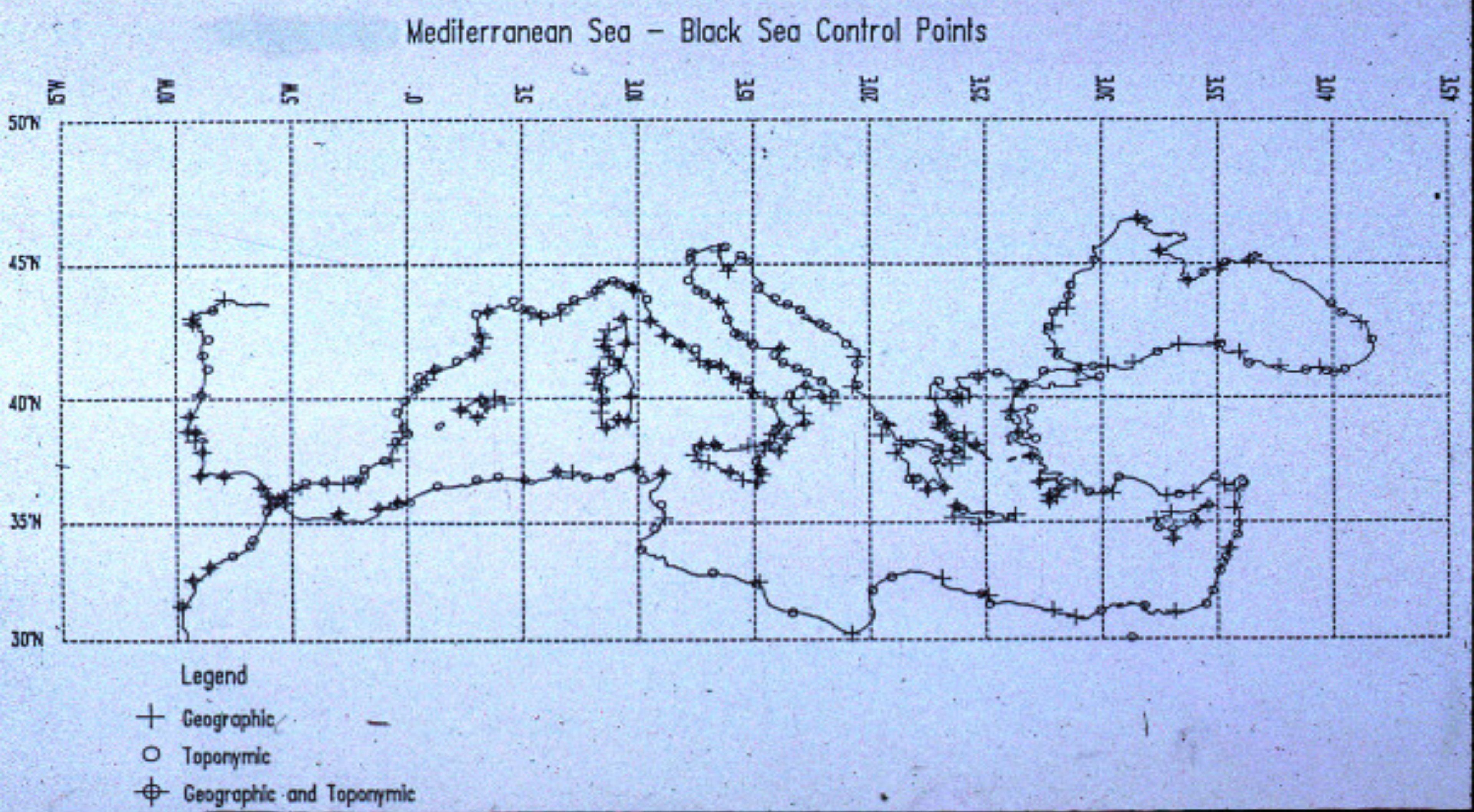
Mediterranean Sea – Black Sea Control Points

Num	Typ	Bas	Lat	Lon	Modern Name / Chart Name(s)[1]
Atlantic Coast, Northwest Spain to Gibraltar, 20 points					
1.005	G		43.77	-7.90	Cabo Ortegal
1.010	T		43.37	-8.40	La Coruna / Corogna, Collogna
1.020	T		43.10	-9.17	Mugia / Mongia
1.025	G		43.07	-9.32	Cabo Tourinan
1.030	B		42.87	-9.27	Cabo Finisterre / Finisterra
1.040	T		42.28	-8.60	Redondela / Radondella, Reondella
1.050	T		41.68	-8.83	Viana do Castello / Ulena, Viena
1.060	T		41.15	-8.62	Porto / Porto Gallo
1.062	B		40.18	-8.90	Cabo Mondego / Mondego
1.065	B		39.35	-9.40	Cabo Carvoeiro / C. Carbon
1.068	B		38.77	-9.50	Cabo da Roca / Roca
1.070	B		38.73	-9.13	Lisboa / Lisbona
1.073	G		38.40	-9.22	Cabo Espichel
1.080	B		37.97	-8.88	Cabo de Sines / Sines, Signes
1.090	B		37.02	-8.98	C. de Sao Vincente / S. Uicenzo
1.100	T		37.02	-7.93	Faro / Faraum, Faran
1.103	G		36.97	-7.92	Cabo de Sta. Maria
1.105	B		36.53	-6.30	Cadiz / Cadis, Cades
1.107	B		36.18	-6.03	Cabo Trafalgar/ Trafagar, Trafalcar
1.110	B		36.02	-5.60	Punta Marroqui / Tarifa
Northwestern Mediterranean, Gibraltar to France, 31 points					
2.010	B	1	36.15	-5.35	Gibraltar / Gibeltar, Zubeltar
2.020	T	1	36.52	-4.88	Marbella / Marbela

[1]The chart names listed do not cover all variants but are representative of the variation in the spelling of the placename.

Mediterranean Nodes

From Loomer



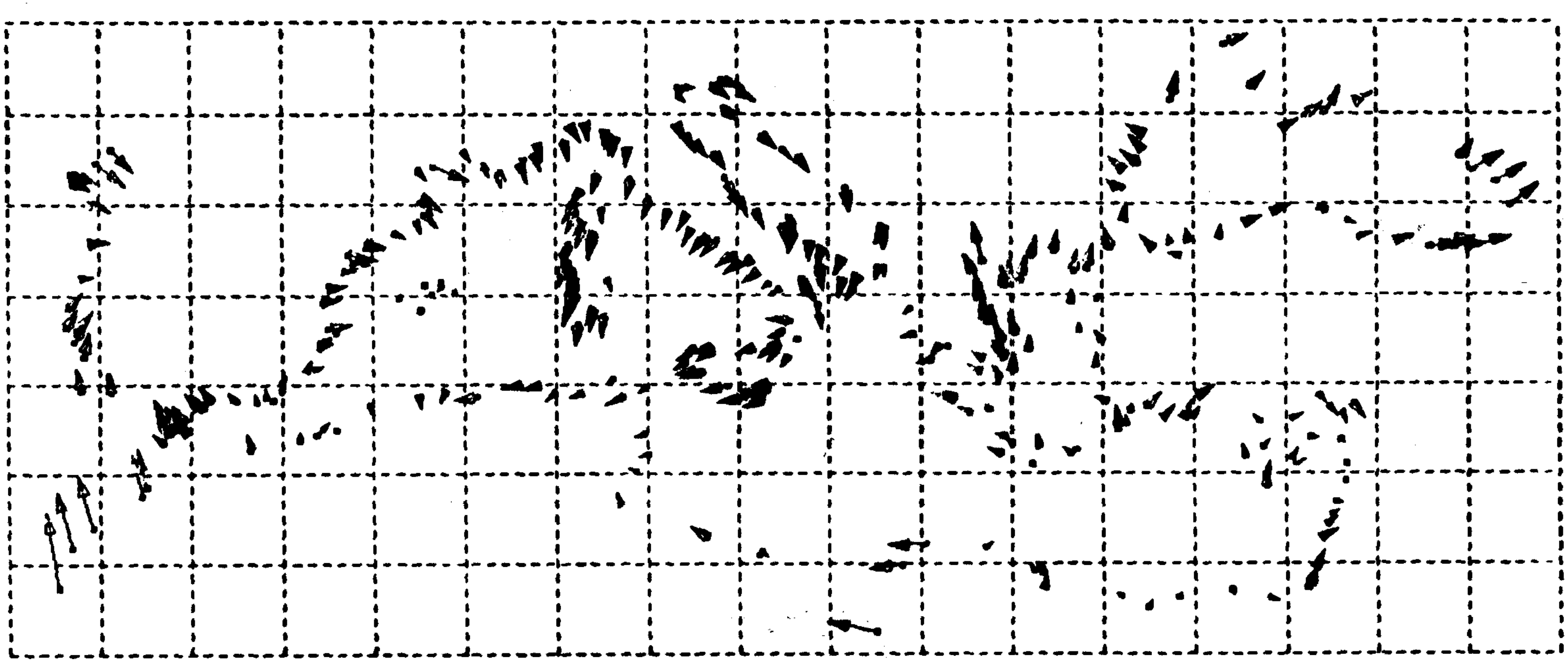
Benincasa 1482

332 Observations

-6.14	43.77	58.66	98.69	1
-6.53	43.37	58.23	97.58	2
-7.13	43.10	56.42	97.37	3
-7.24	43.07	55.85	97.47	4
-7.20	42.87	55.85	96.82	5
-6.68	42.25	57.54	95.56	6
-6.70	41.15	57.80	93.43	7

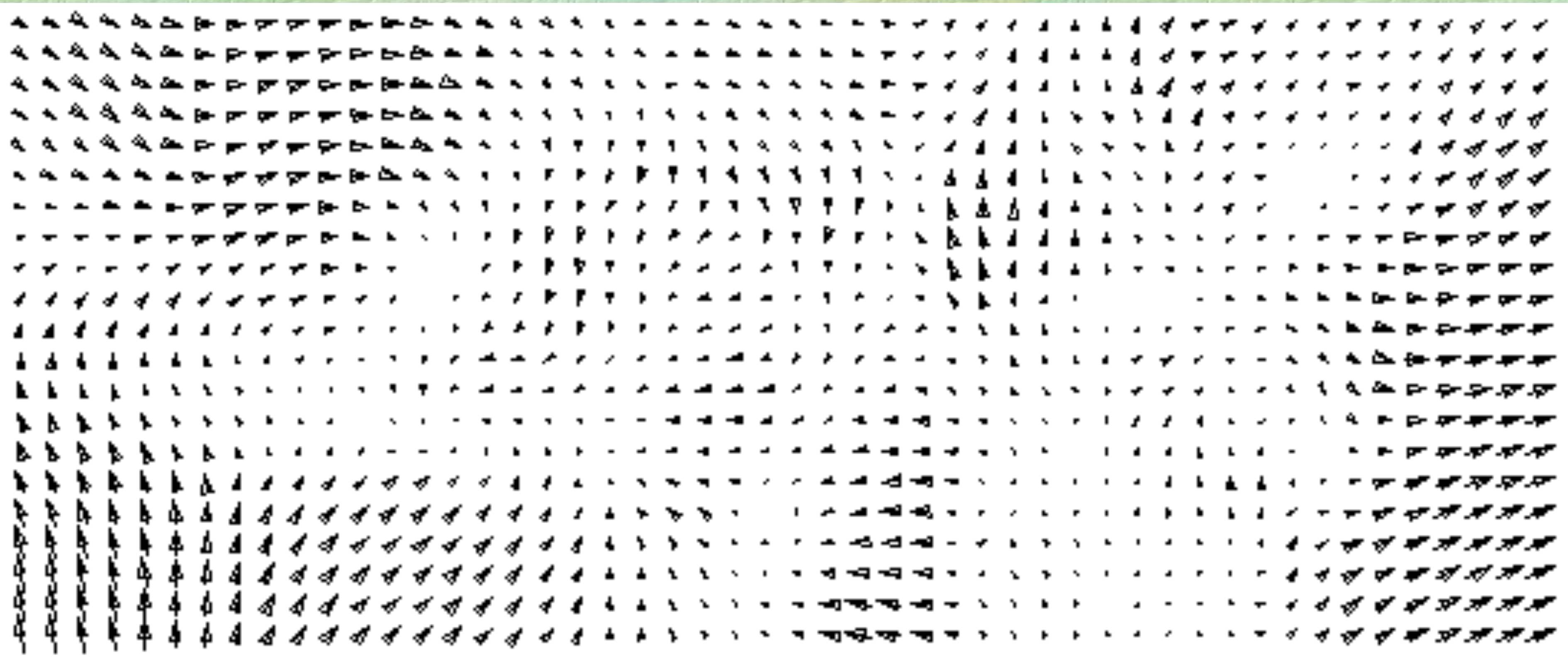
. . . .

Mediterranean Displacements



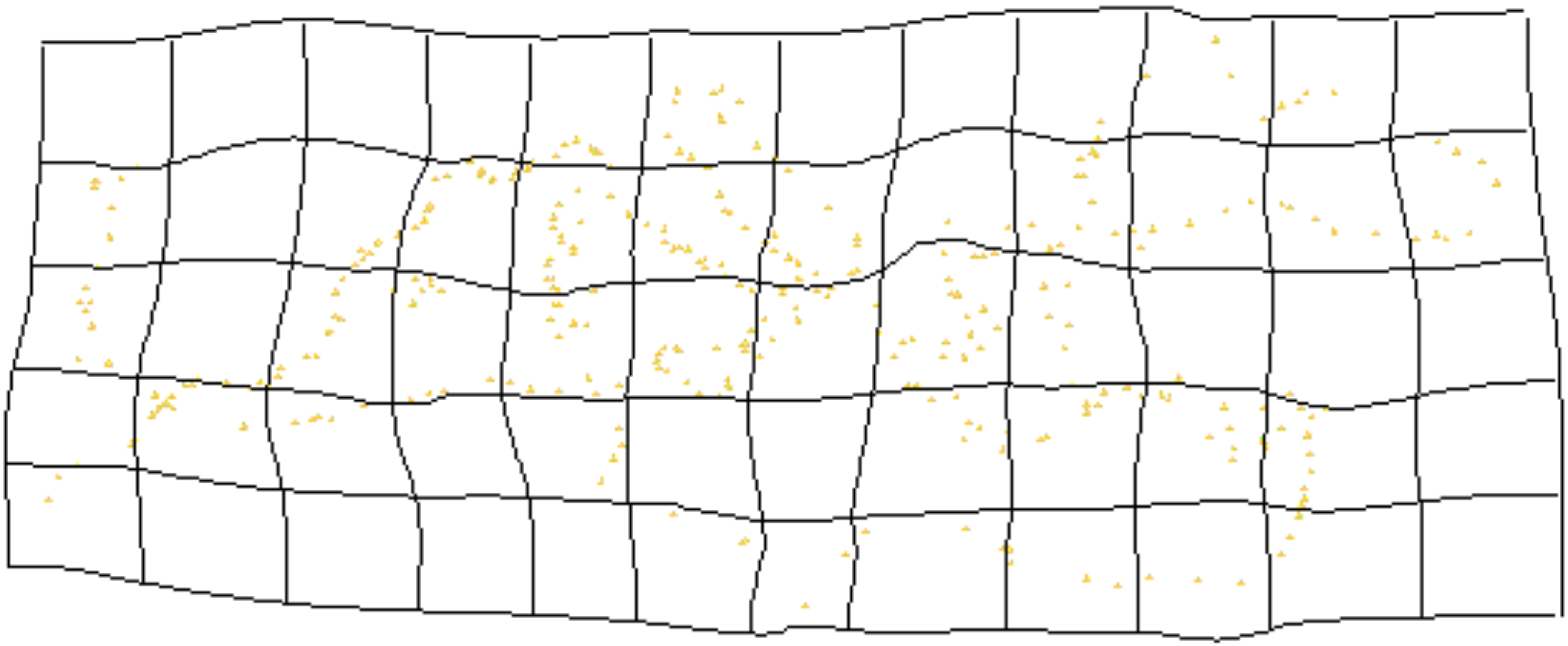
Interpolated Vector Field

Based on Mediterranean displacements



Warped Grid of Portolan Chart

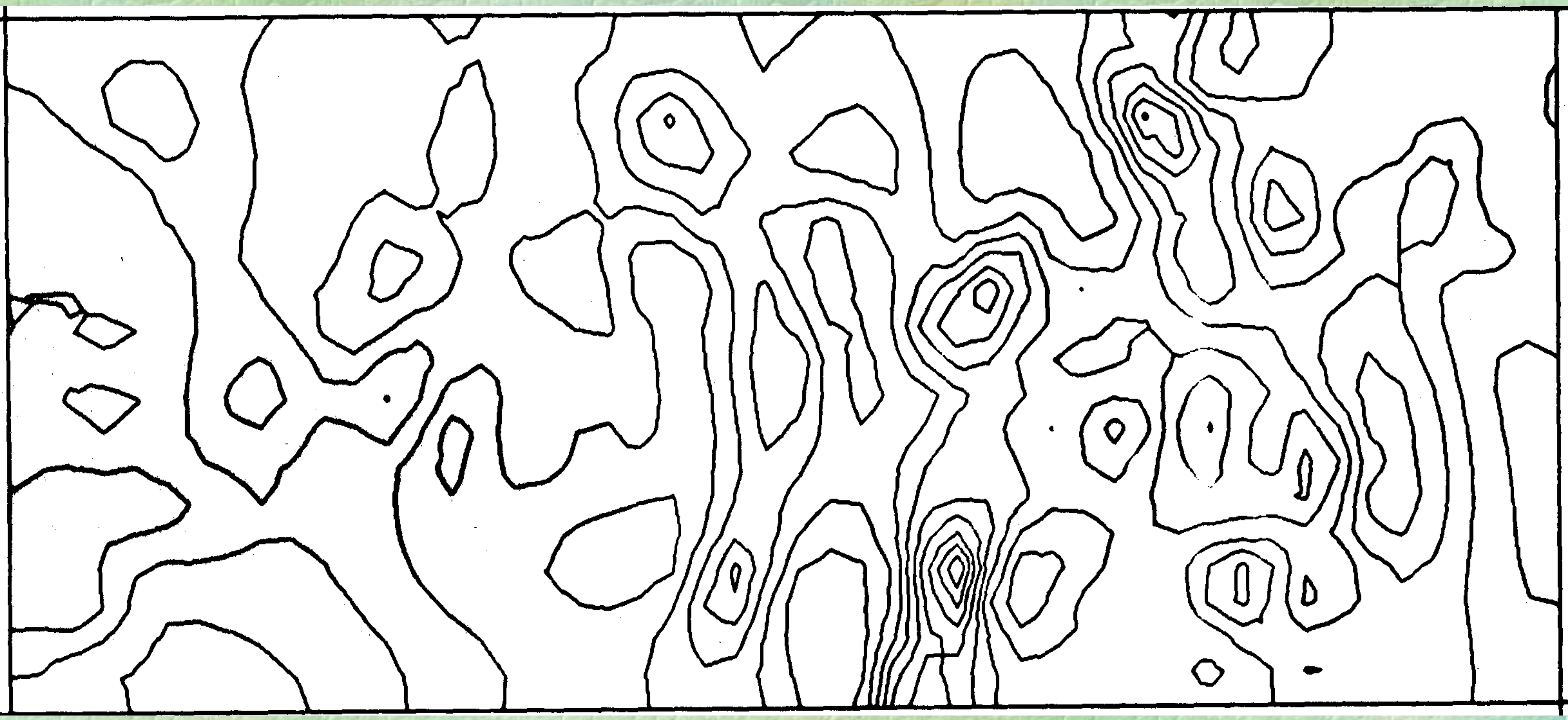
As 'pushed' by the interpolated vector field



A simple measure of total distortion at each point is the sum of squares of the partial derivatives.

This may also be applied to the rubber sheeting shown earlier, or to the migration maps shown later, although in this case the interpretation is more difficult.

Total Distortion on the 1482 Portolan



Tissot's Indicatrix also Measures distortion

It is based on the four partial derivative of the transformation, $\delta u/\delta x$, $\delta v/\delta x$, $\delta u/\delta y$, $\delta v/\delta y$.

As such it is a tensor function of location. It varies from place to place, and reflects the fact that **map scale is different in every direction** at a location, unless the map is conformal.

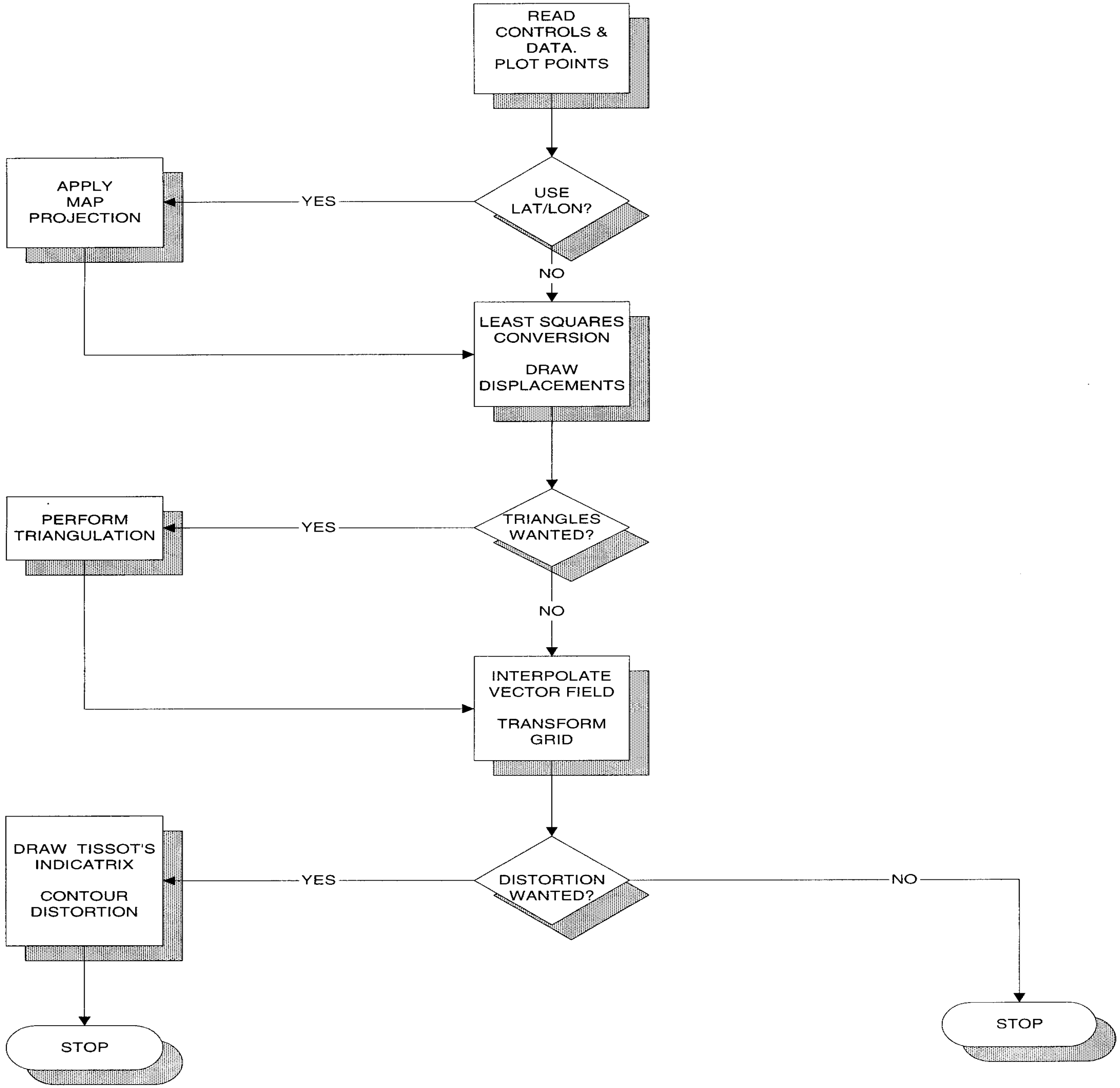
The Coastlines May be Drawn Using the Warped Grid

Observe that either the old map, or the modern one, can be considered the independent variable in this bidimensional regression.

Relating two sets of coordinates (the old and the new) requires a bidimensional correlation, instead of a regular unidimensional correlation, as did the relation between the student map coordinates and the actual coordinates. The bidimensional correlation can be linear or curvilinear.

W. Tobler, 1994, "Bidimensional Regression", *Geographical Analysis*, 26 (July): 186-212

BIDIMENSIONAL REGRESSION PROGRAM



Waldo Tobler

Asymmetrical Tables Can Also Lead To Construction Of A Vector Field

Start with an asymmetrical geographical table. There are many such tables!

It is possible to compute the degree of asymmetry for such tables, and to partition the total variance into symmetric and skew symmetric variances

To construct the vector field it is necessary to know the geographic locations and to invoke a model of the process.

An example of an asymmetric geographical table.

Polynesian Communication Charges ($)

From \ To	CI	Fiji	F P	Ki	N C	PNG	SI	To	Tu	Va	WS	Aust	Fr	Ja	NZ	UK	USA
Cook I		5.29	5.29	5.29	5.29	5.29	5.29	5.29	5.29	5.29	5.29	5.29	9.35	9.35	3.70	9.35	9.35
Fiji	3.29		3.50	3.29	3.50	3.29	3.29	3.29	na	3.50	3.29	3.29	5.55	5.55	3.29	5.55	5.55
F Polynesia	14.85	14.85		na	7.95	14.85	14.85	14.85	na	7.95	14.85	12.73	15.91	24.39	12.73	21.21	24.39
Kiribati	6.50	6.50	6.50		6.50	6.50	6.50	6.50	6.50	6.50	6.50	6.50	9.28	9.28	6.50	9.28	9.28
N Caledonia	13.85	13.85	7.79	na		11.70	11.70	13.85	na	7.79	11.70	11.70	15.61	17.53	11.70	19.91	23.36
Papua N G	5.67	5.67	5.67	5.67	5.67		5.67	5.67	5.67	5.67	5.67	5.67	11.33	11.33	5.67	7.55	11.33
Solomon I	5.30	4.12	5.30	5.30	5.30	4.12		5.30	5.30	5.30	5.30	3.82	8.53	8.53	4.12	8.53	8.53
Tonga	3.47	3.47	3.47	3.47	3.47	3.47	3.47		3.47	3.47	3.47	3.47	6.95	6.95	3.47	6.95	6.95
Tuvalu	5.80	4.64	5.80	5.80	5.80	5.80	5.80	5.80		5.80	5.80	3.48	9.28	9.28	9.28	9.28	9.28
Vanuatu	7.96	5.24	7.96	7.96	5.24	7.96	7.96	7.96	7.96		7.96	5.24	10.29	10.29	5.24	10.29	13.02
W Samoa	3.86	3.86	3.86	3.86	3.86	3.86	3.86	3.86	3.86	3.86		3.86	5.14	5.14	3.86	5.14	5.14
Australia	3.71	3.71	3.71	3.71	3.71	3.02	3.02	4.87	na	3.71	3.71		3.71	4.87	3.02	3.71	3.71
France	12.80	12.80	7.53	na	7.53	12.80	12.80	12.80	na	12.80	12.80	11.17		11.17	11.17	2.63	5.47
Japan	6.27	6.27	6.27	6.27	6.27	6.27	6.27	6.27	na	6.27	6.27	6.27	7.88		6.27	7.88	5.20
N Zealand	2.79	2.79	2.79	2.79	2.79	2.79	2.79	2.79	2.79	2.79	2.79	2.79	5.40	5.40		5.40	5.40
UK	8.21	8.21	8.21	8.21	8.21	8.21	8.21	8.21	8.21	8.21	8.21	5.84	2.66	8.21	5.84		4.11
USA	5.77	5.77	5.49	5.77	5.77	5.77	5.77	5.72	5.77	5.77	5.72	4.66	5.27	4.14	5.53	4.08	

R.G. Ward, 1995, “The Shape of the Tele-Cost Worlds”, A. Cliff, et al, eds., *Diffusing Geography*, p. 228.

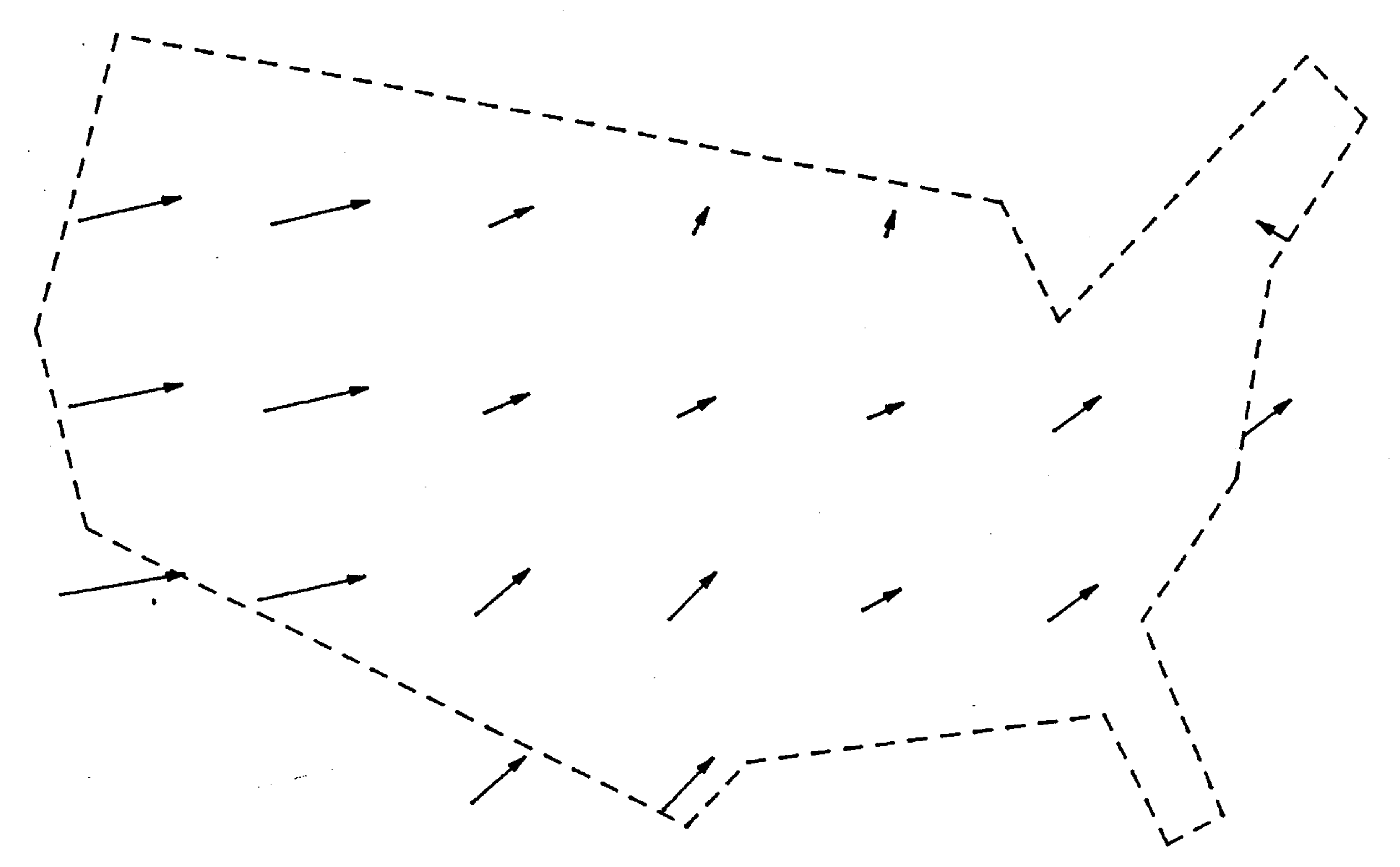
Another example

Table of Mail Delivery Times

Transit time for US mail, in days (1973)

From: \ To:	NYC	CHI	LAX	WDC	STL	HOU
NYC	0.9	1.8	2.5	2.0	2.3	2.3
CHI	2.6	0.8	3.1	2.2	1.9	2.3
LAX	2.5	2.2	1.1	2.2	2.3	2.6
WDC	1.8	2.3	2.6	1.3	2.4	2.5
STL	2.4	2.1	3.1	2.4	0.9	2.5
HOU	2.3	1.9	2.8	2.2	2.2	1.1

Wind Pattern Computed From Mail Delivery Time



One of the Interesting Things About Vector Fields Is That They Can Be Inverted.

That is, given the slope of a topography, one can compute the elevations, up to a constant of integration.

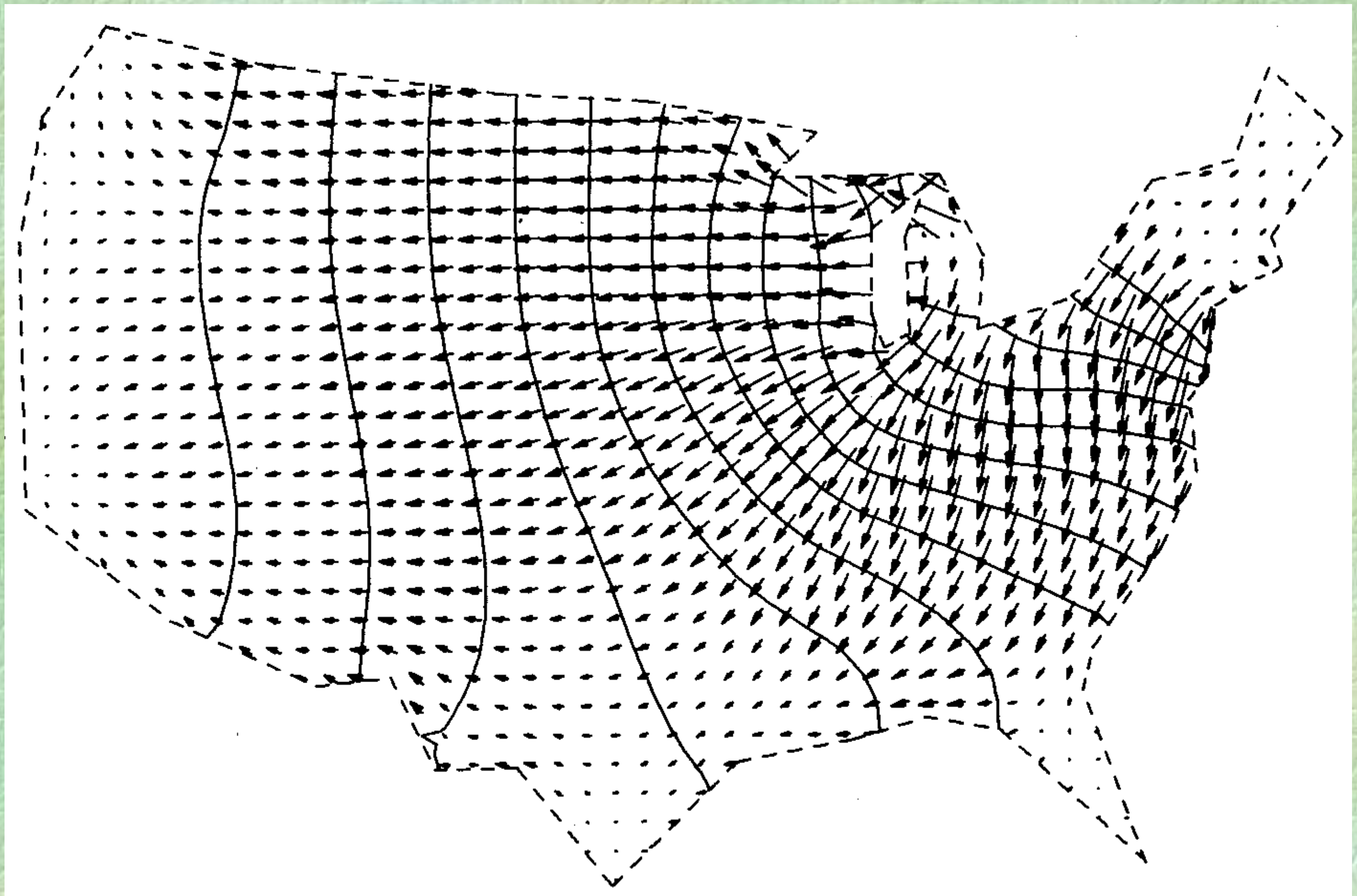
So, for example, the implied pressure field for the previous wind field could be computed.

This assumes that the vector field is curl free.

Another Example

Where the Government Spends Your Money

Fiscal Transfers via Federal Accounts



Do you feel that you get your share? The contours show the implied political “pressure”. The vectors show the estimated movement of funds.

W. Tobler, 1981, “Depicting Federal Fiscal Transfers”, *Professional Geographer*, 33(4):419-422.

Migration Data Often Come in the Form of Square Tables

The rows represent the “from” places and the columns the “to” places.

The tables are not symmetrical!

A Nine Region US Migration Table

Observe that it is not symmetric!

Thus there will be places of depletion and places of accumulation!

Nine Region Migration Table

US Census 1973

	1	2	3	4	5	6	7	8	9
1 New England	—	180,048	79,223	26,887	198,144	17,995	35,563	30,528	110,792
2 Mid-Atlantic	283,049	—	300,345	67,280	718,673	55,094	93,434	87,987	268,458
3 East North Central	87,267	237,229	—	281,791	551,483	230,788	178,517	172,711	394,481
4 West North Central	28,977	60,681	286,580	—	143,860	49,892	185,618	181,868	274,629
5 South Atlantic	130,830	382,565	346,407	92,308	—	252,189	192,223	89,389	279,739
6 East South Central	21,434	53,772	287,340	49,828	316,650	—	141,679	27,409	87,938
7 West South Central	30,287	64,645	161,645	144,980	199,466	121,366	—	134,229	289,880
8 Mountain	21,450	43,749	97,808	113,683	89,806	25,574	158,006	—	437,255
9 Pacific	72,114	133,122	229,764	165,405	266,305	66,324	252,039	342,948	—

This is an example of a census migration table. There are also (50 by 50) state tables and county by county tables.

There is a great deal of spatial coherence in the migration pattern

In the US case the state boundaries hide the effect, as would the county boundaries in the UK case. Therefore they are omitted.

There is also temporal coherence.

W. Tobler, 1995, "Migration: Ravenstein, Thornthwaite, and Beyond", *Urban Geography,* 16(4):327-343.

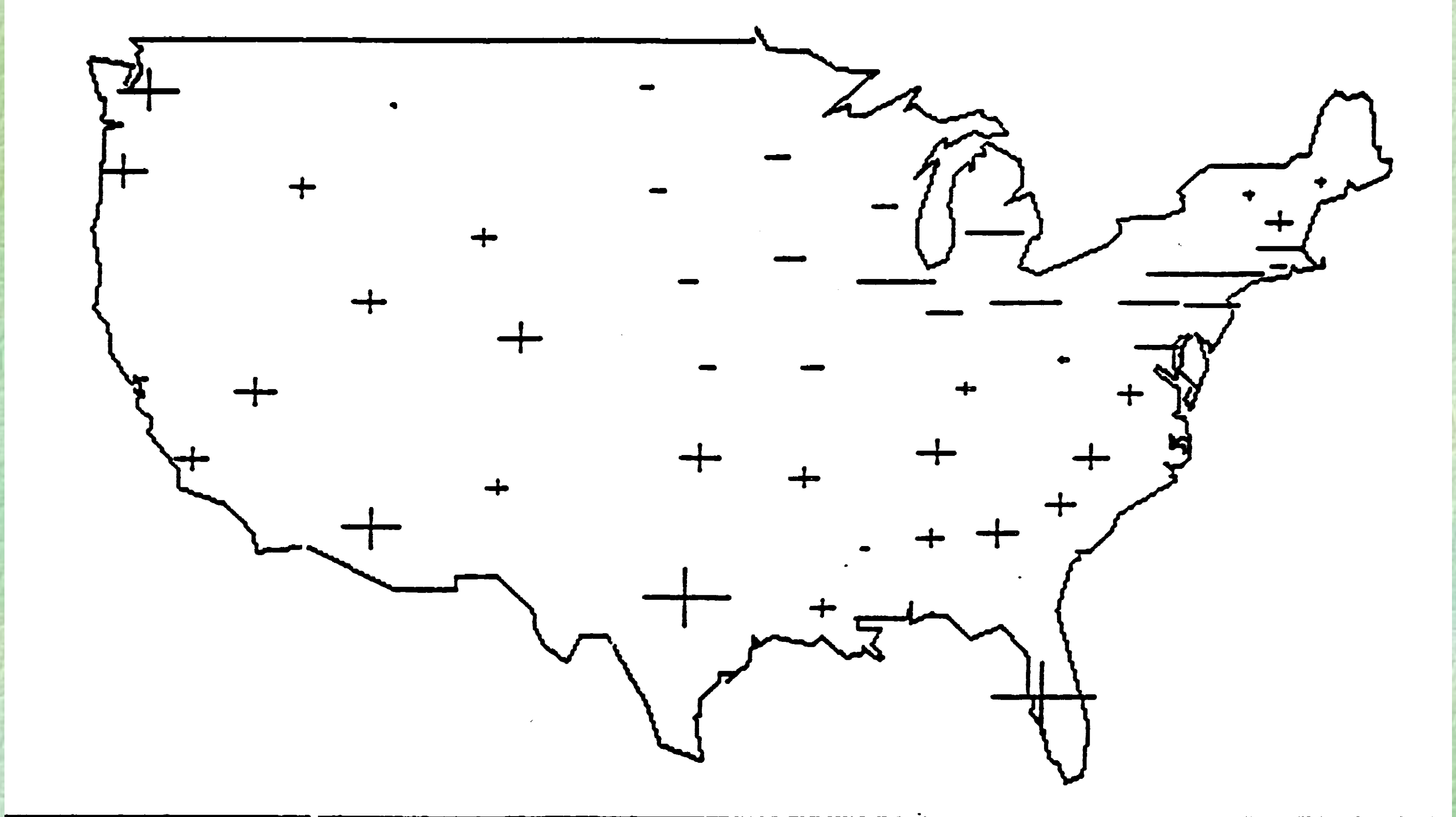
Gaining and Losing States

Symbol positioned at the state centroids, and proportional to magnitude of the change.

Migration in the United States

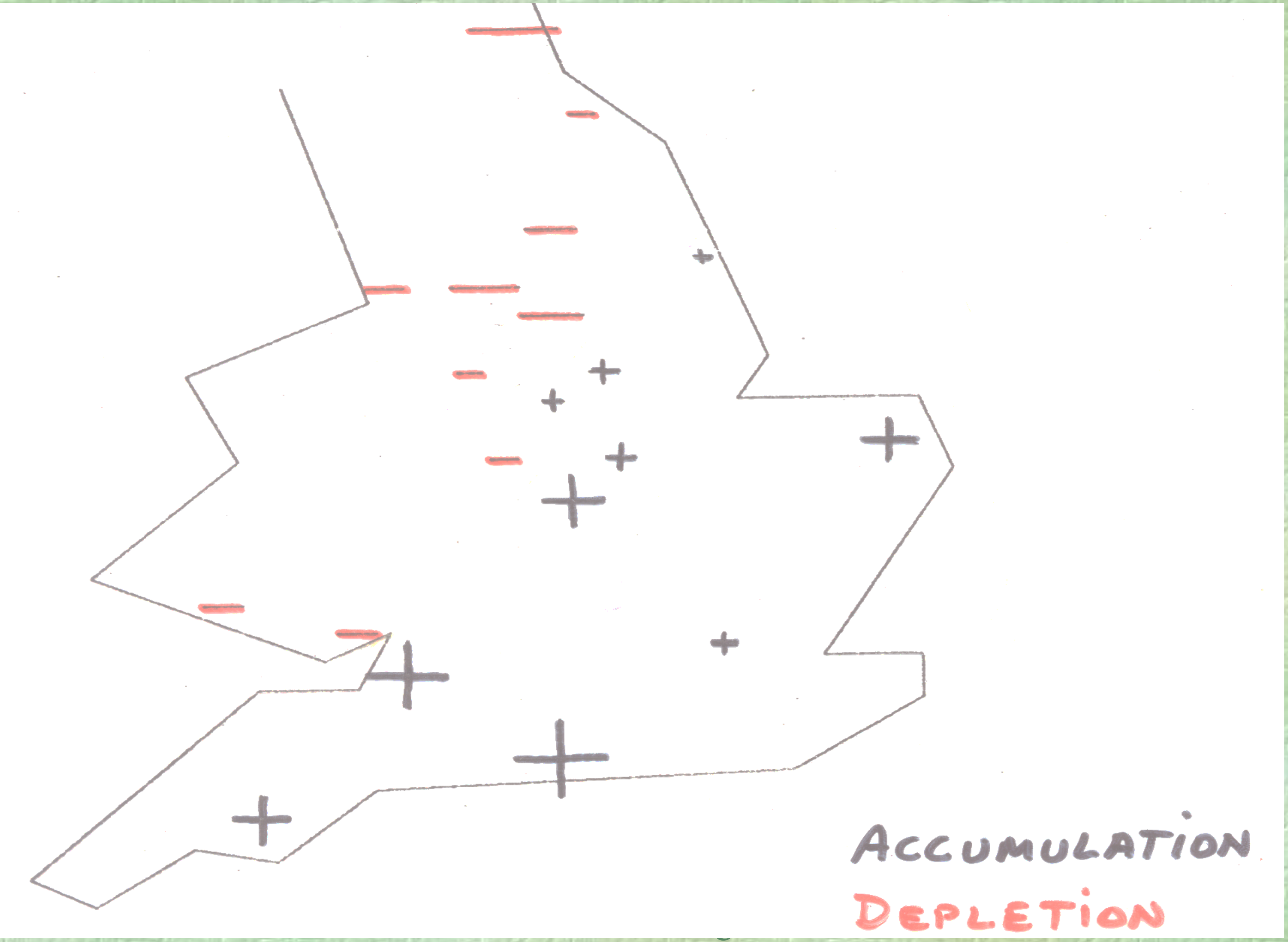
The map is based on the marginals of a 48 x 48 state to state migration table

and shows the accumulation and depletion places



Net County Migration in England

1960-1961



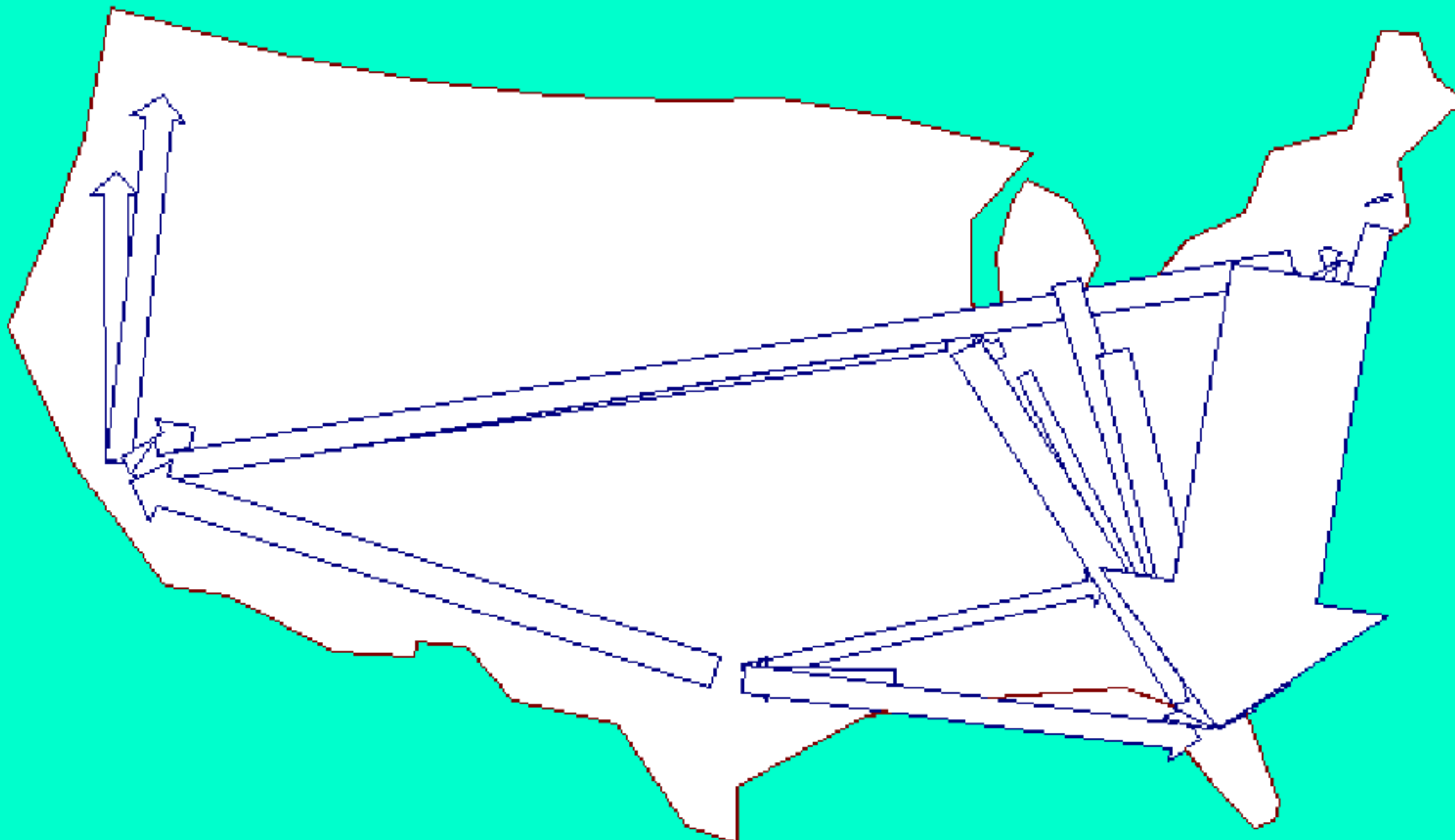
Data from Fielding

Conventional Computer Drawn Flow Map

Major movement shown between state centroids.

Net Movement Shown

The map is based on the marginals of a 48 x 48 state to state migration table.



Notice that only the Net Movements from the Table are being used

These are the difference of the marginals.

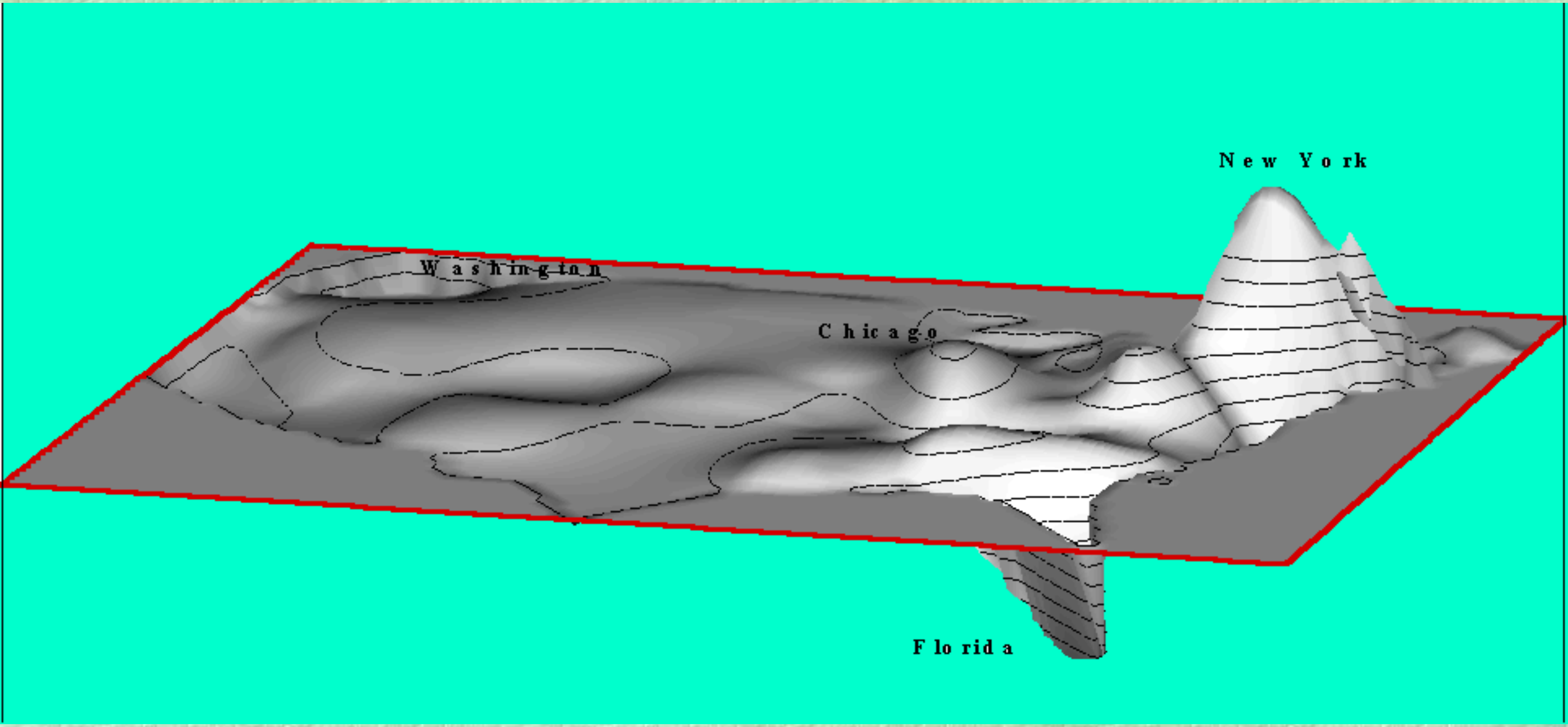
In-movement minus out-movement.

From the asymmetry of the table margins one can compute an attractivity, or pressure to move. Of course this requires a model.

G. Dorigo, & Tobler, W., 1983, "Push-Pull Migration Laws", *Annals*, AAG, 7391):1-17.

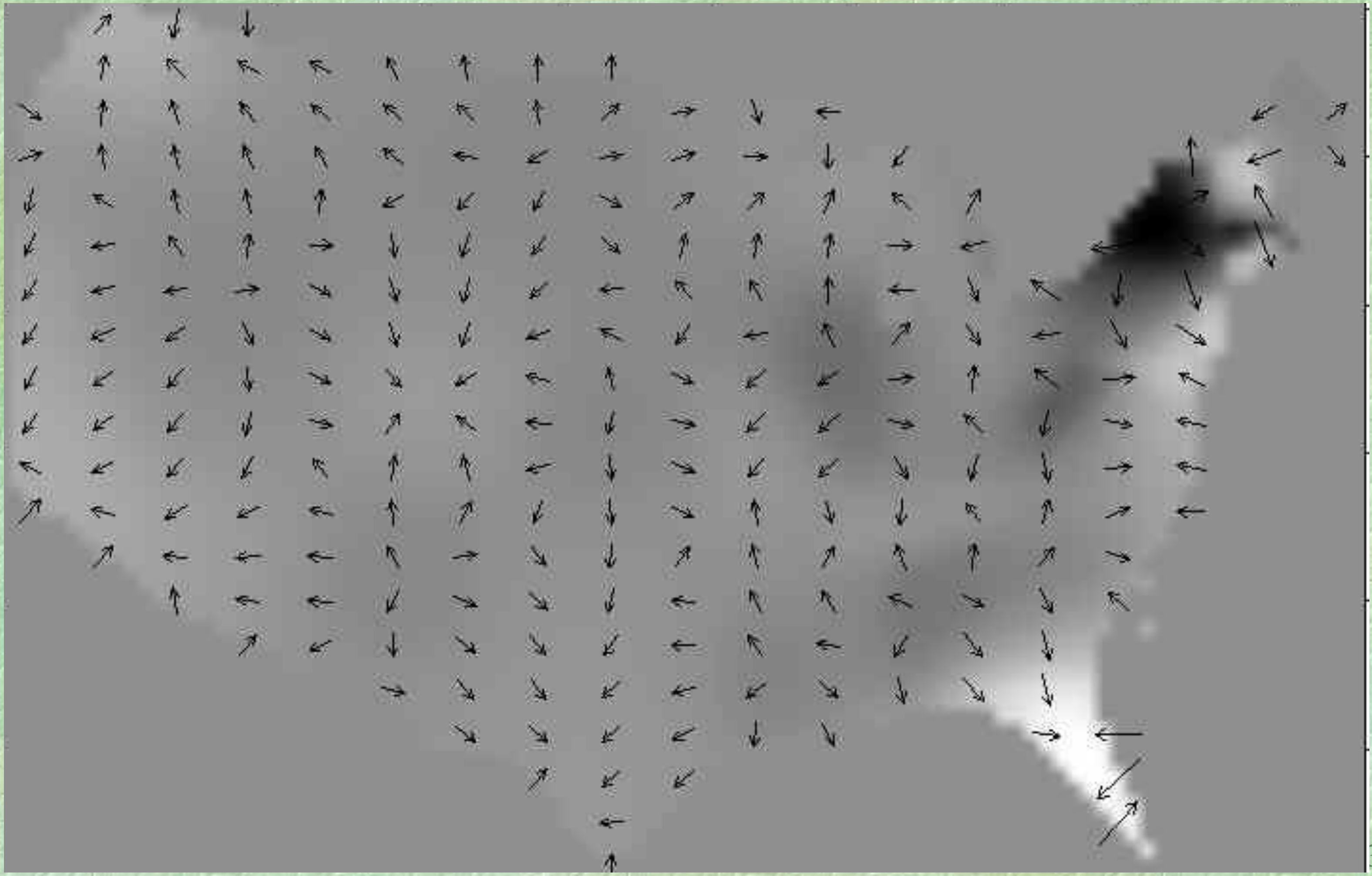
Pressure to Move in the US

Based on a continuous spatial gravity model



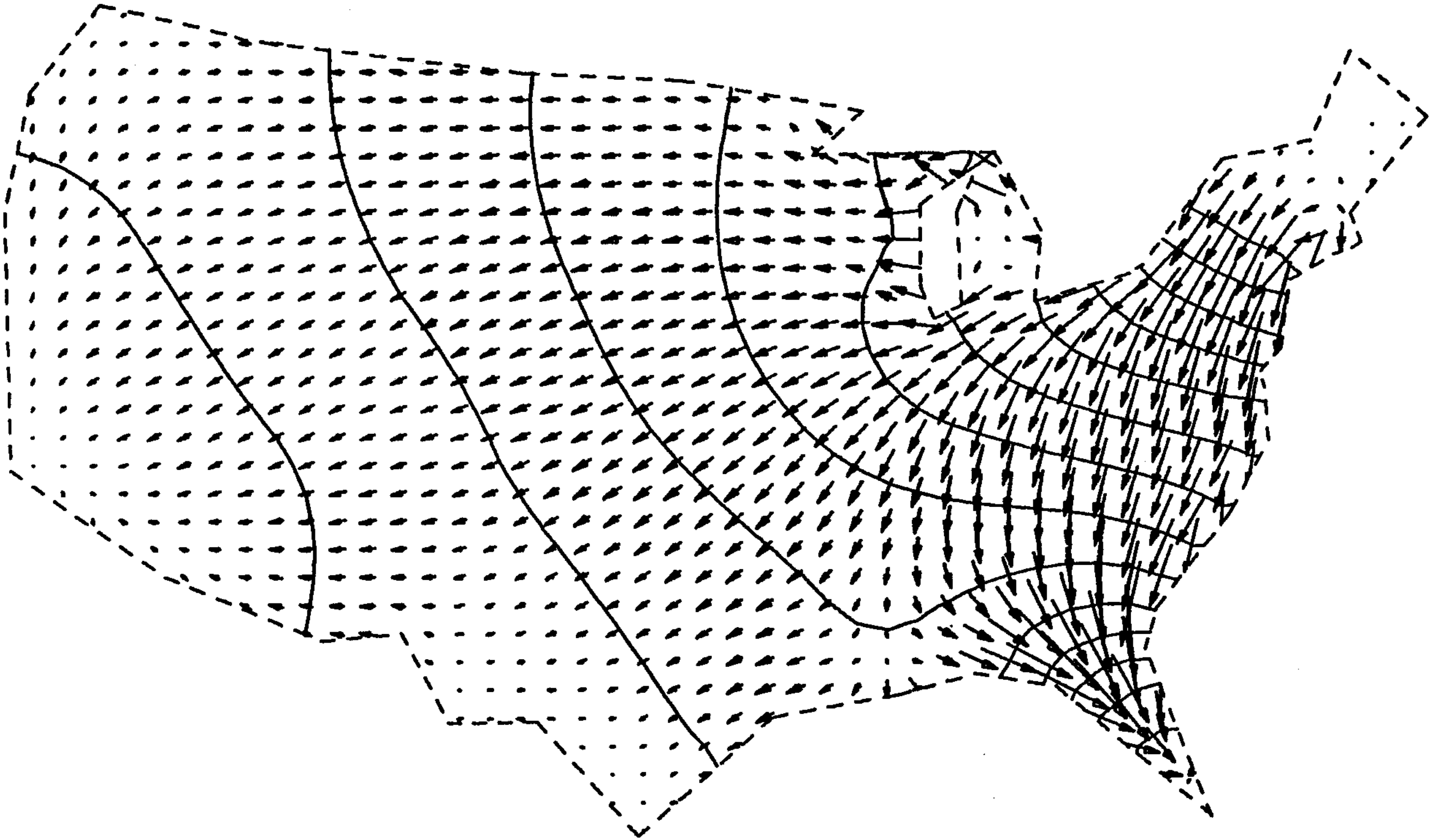
Migration Potential and Gradients

Another view of the same model



Migration Potentials and Gradients

Potentials computed from a continuous gravity model and shown by contours

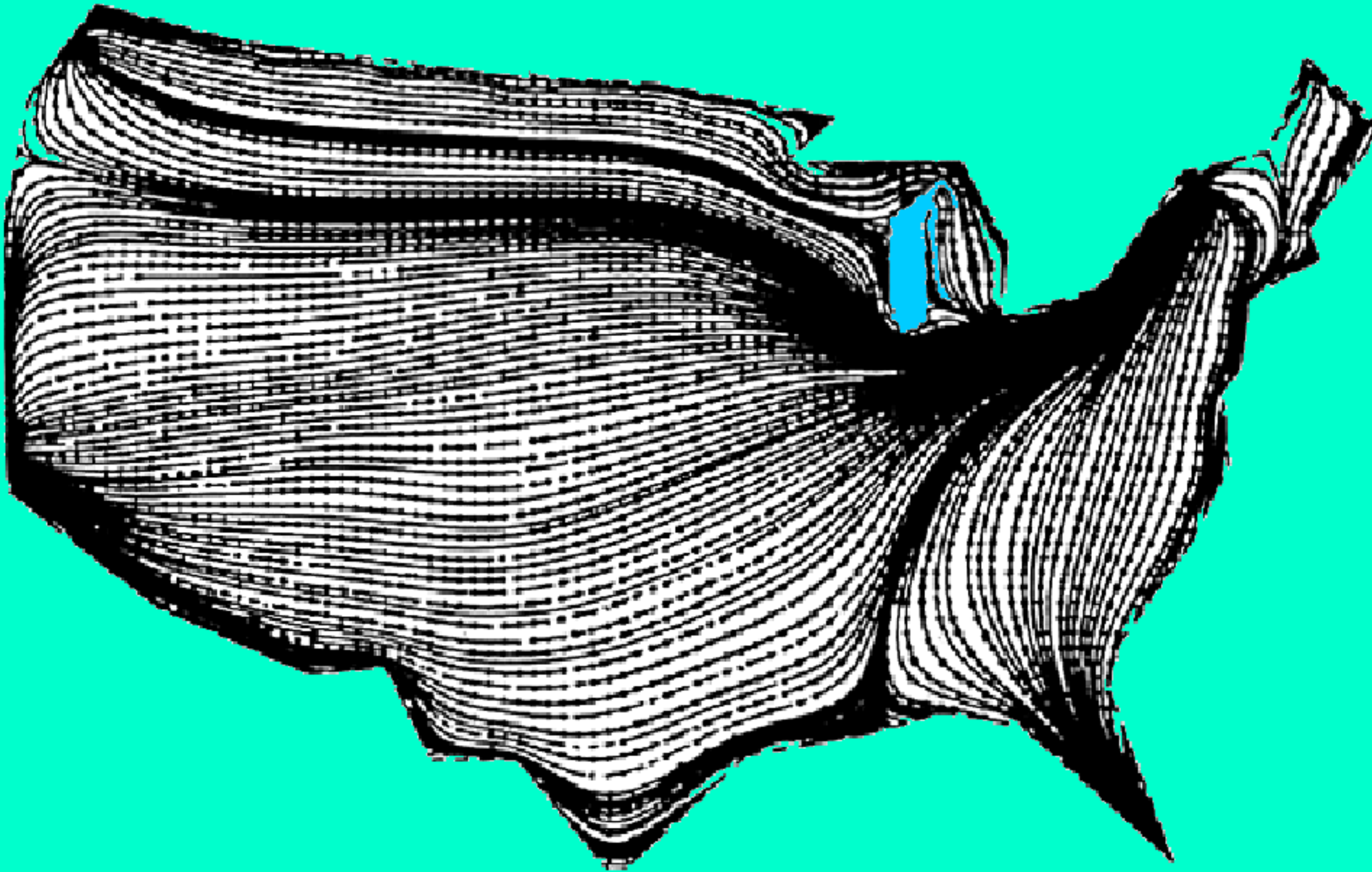


Recall that several million people migrate during the 5 year census period

The next map shows an ensemble average, not the path of any individual.

But observe, not unrealistically, that the people to the East of Detroit tend to go to the Southeast, and Minnesotans to the Northwest, and the remainder to the Southwest.

16 Million People Migrating



Changing the resolution acts as a spatial filter.

This is shown by vector fields at several levels of resolution.

The next several maps are of net migration in Switzerland.

3.6 km resolution (3090 Gemeinde)

14.7 km resolution (184 Bezirke)

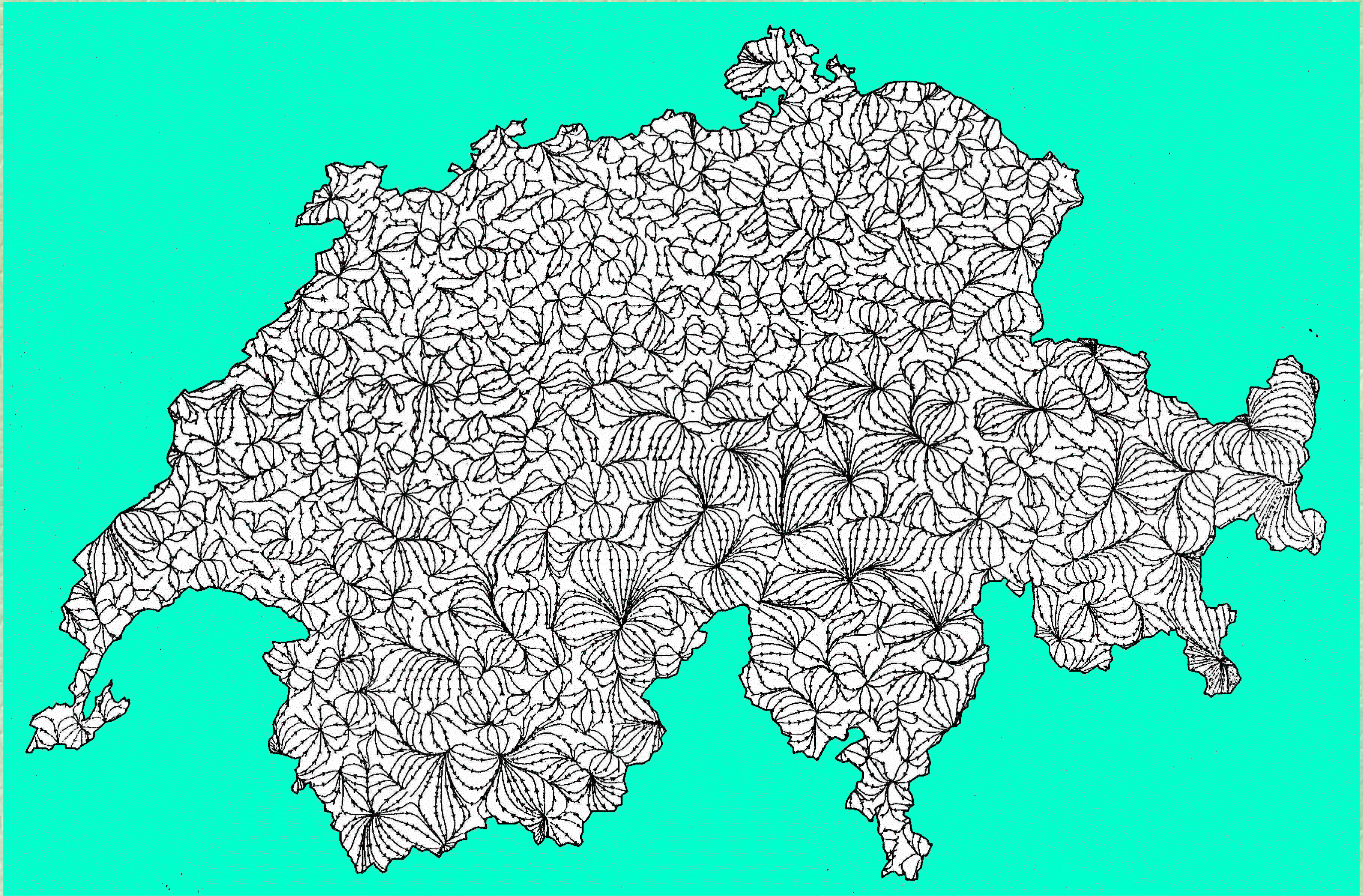
39.2 km resolution (26 Kantone)

Maps by Guido Dorigo, University of Zürich

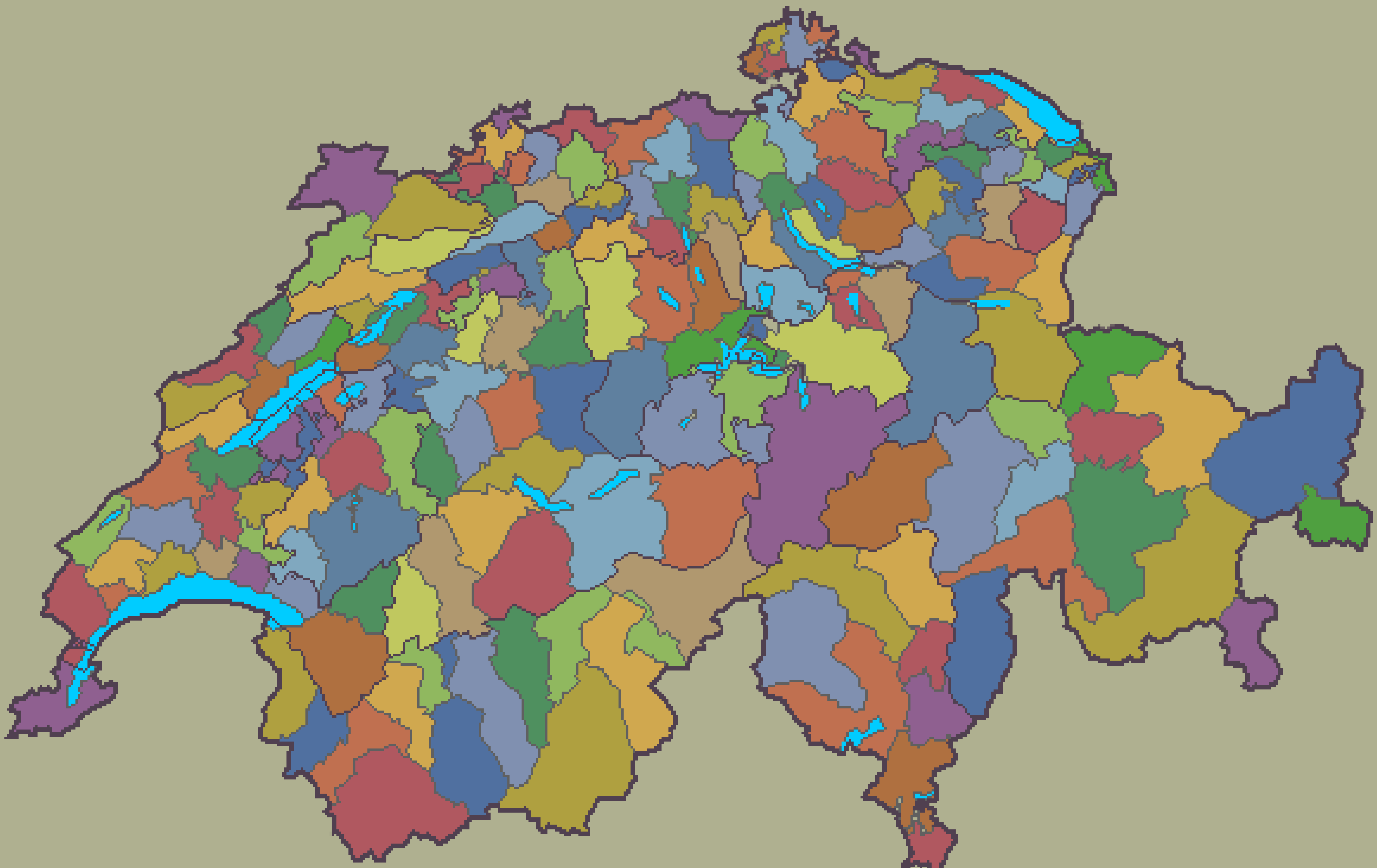
3090 Communities. 3.6 km average resolution

Migration "Turbulence" in the Alps

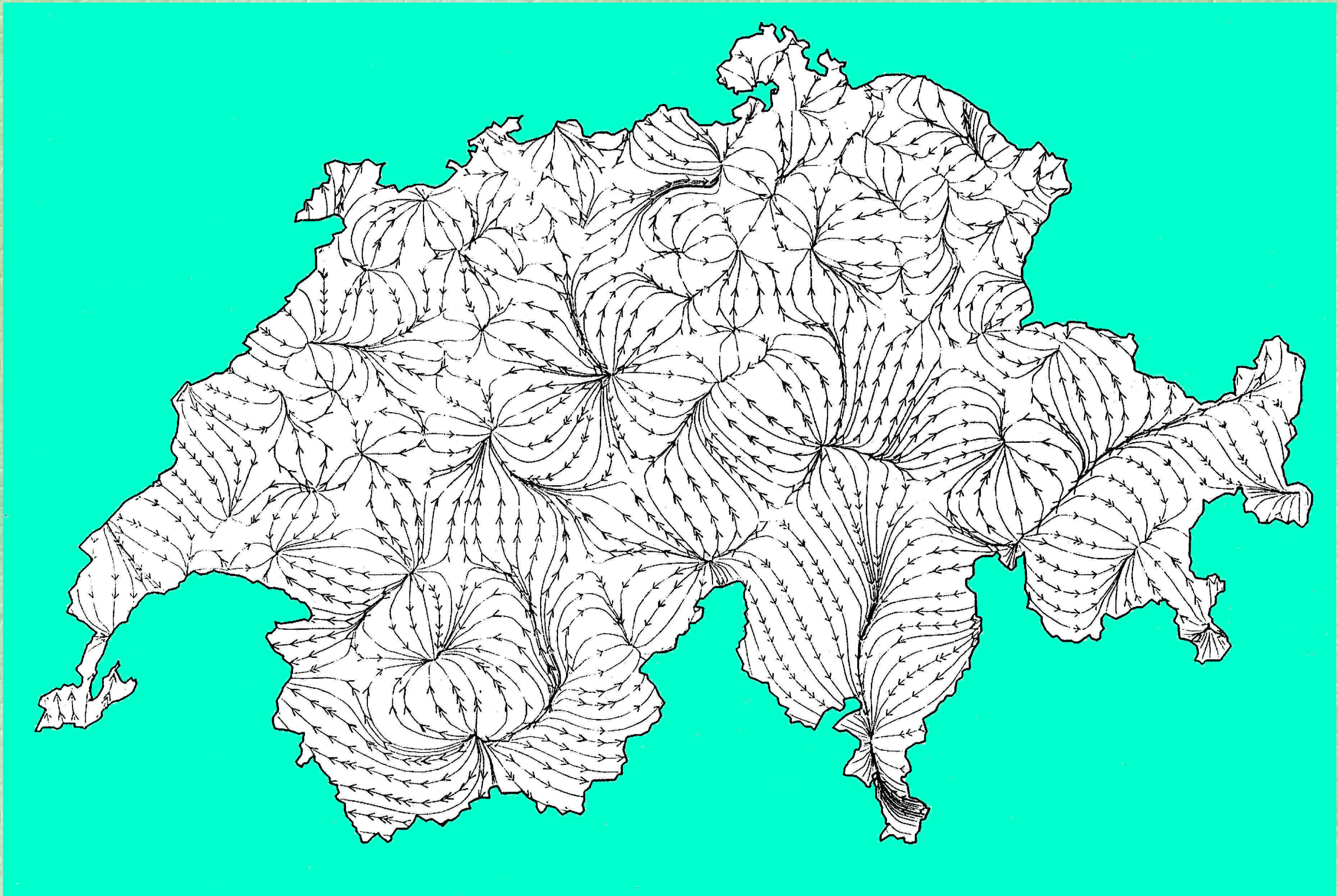
3.6 km resolution



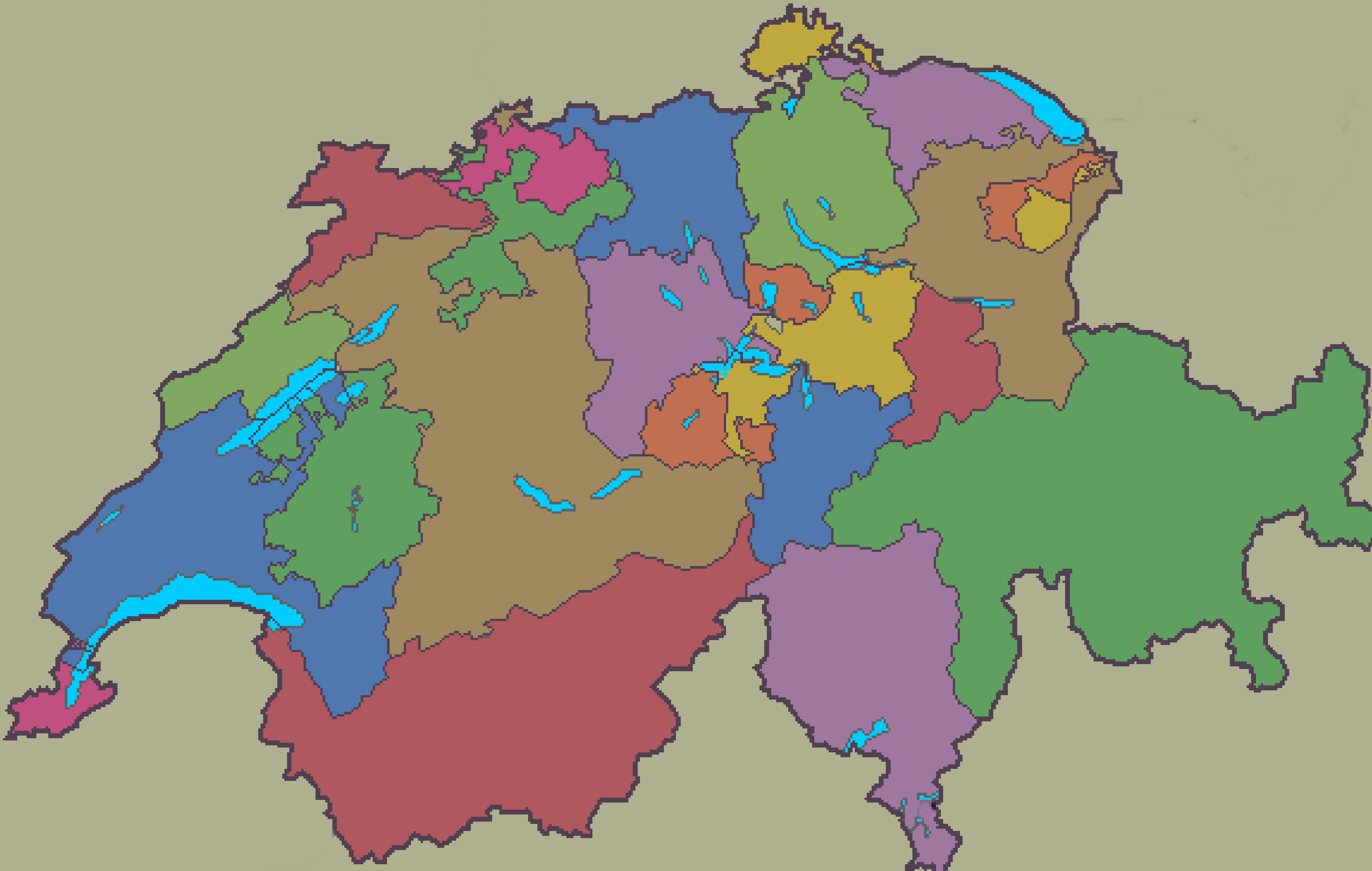
184 Districts. 14.7 km average resolution



Less of the Fine Detail
14.7 km resolution



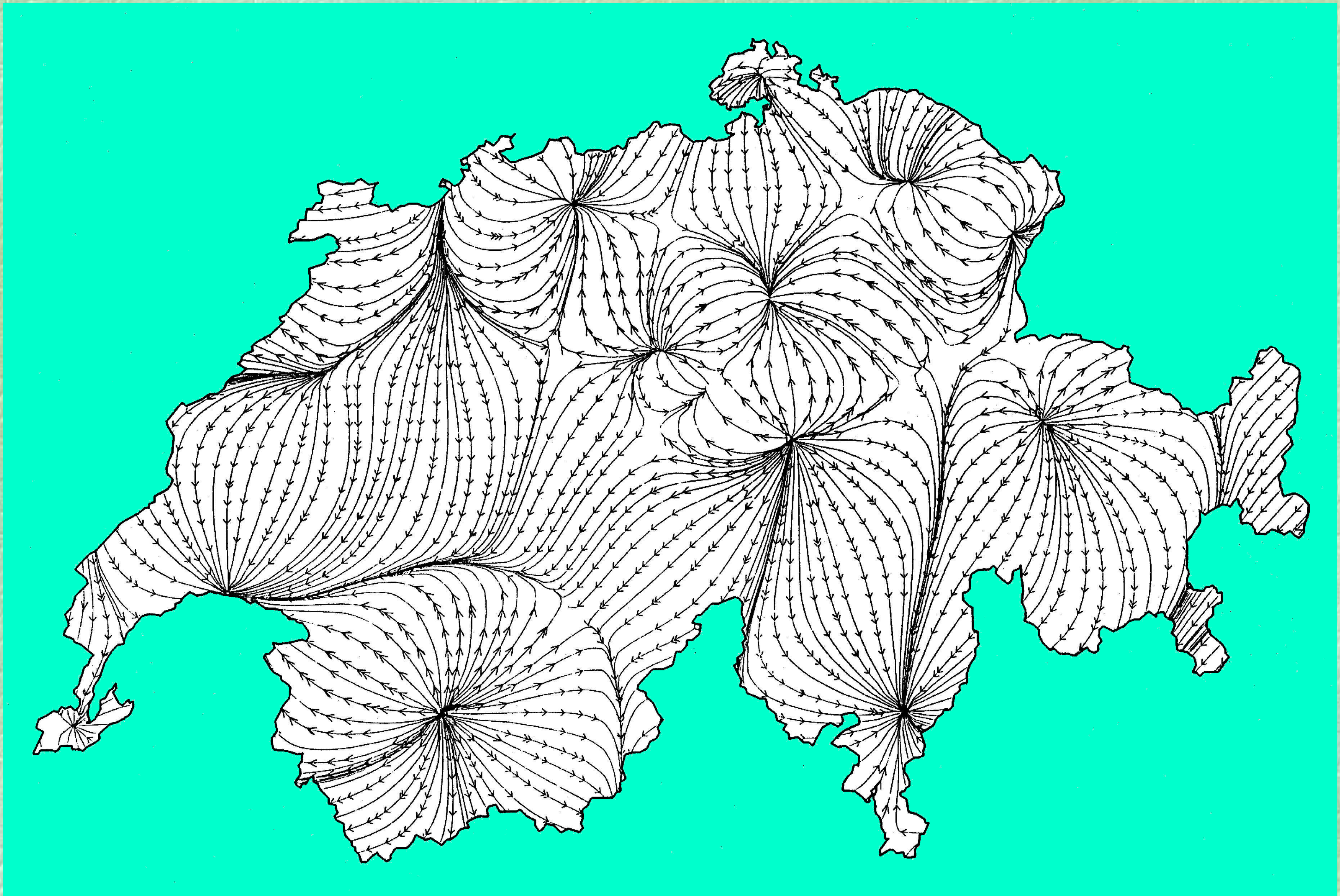
26 Cantons. 39.2 km. average resolution



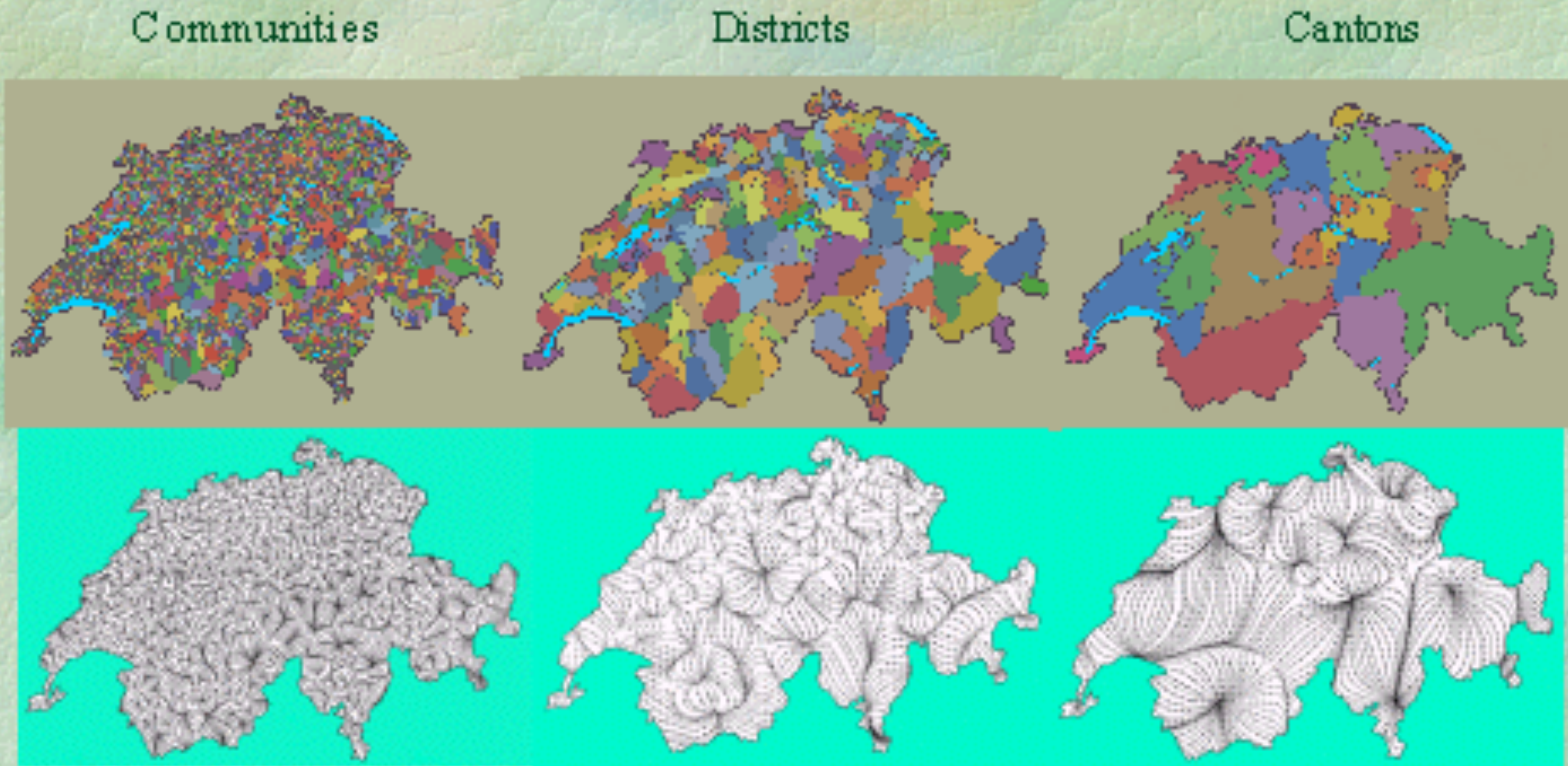
The Broad Pattern Only

39.2 km resolution

Changing the resolution has the effect of a spatial filter.



Three levels of administrative units and three levels of migration resolution all at once.



Some consequences of Resolution for Movement Studies.

A State to State migration table yields a 50 by 50 migration table, with 2,500 entries. Patterns as small as 800 km in extent might be seen.

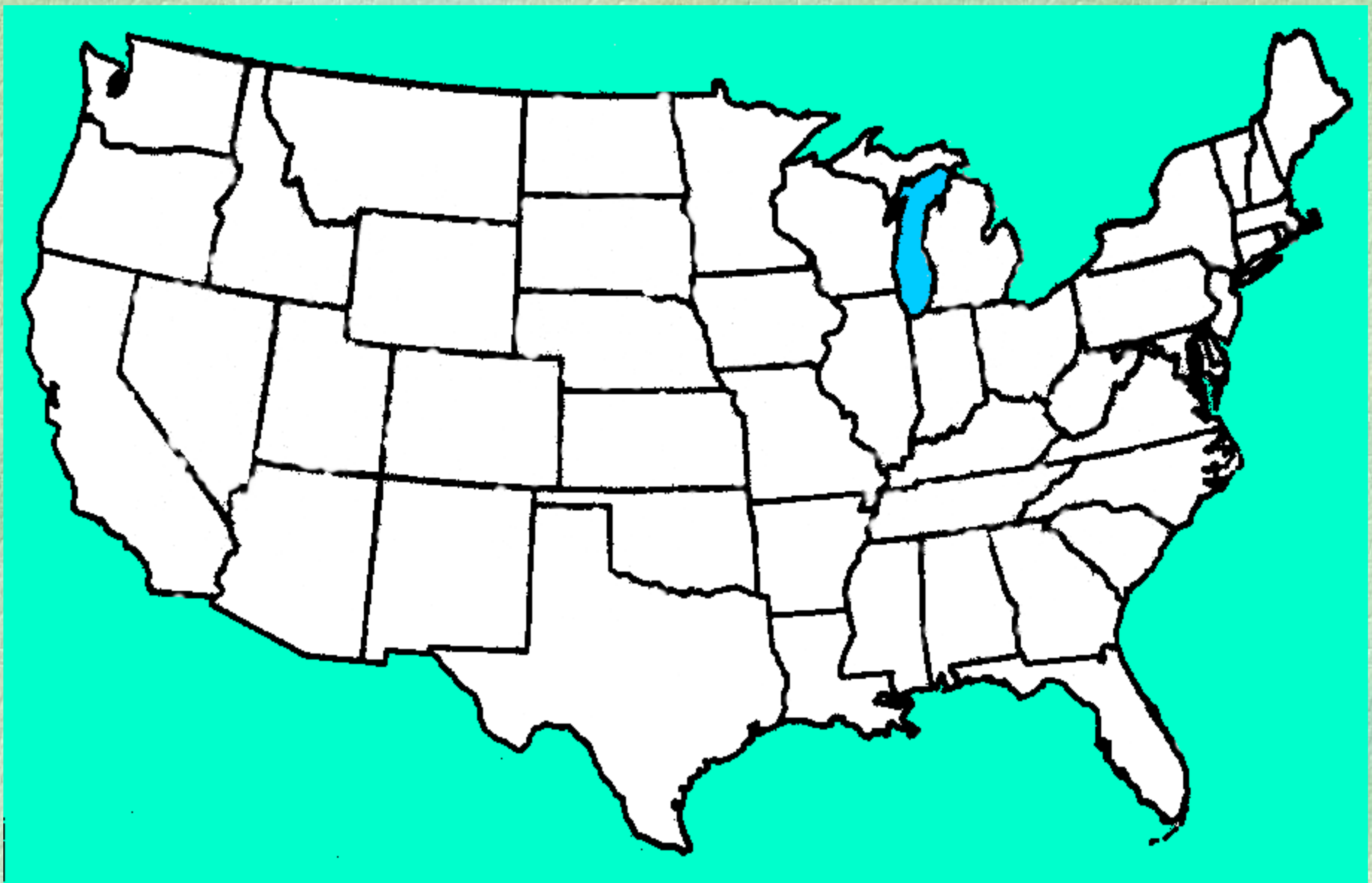
A county to county migration table 3141 by 3141 in size could contain over 9 million entries. (It actually contains only 5% of these).

A table of worldwide movement or trade between all countries could contain nearly 40,000 numbers.

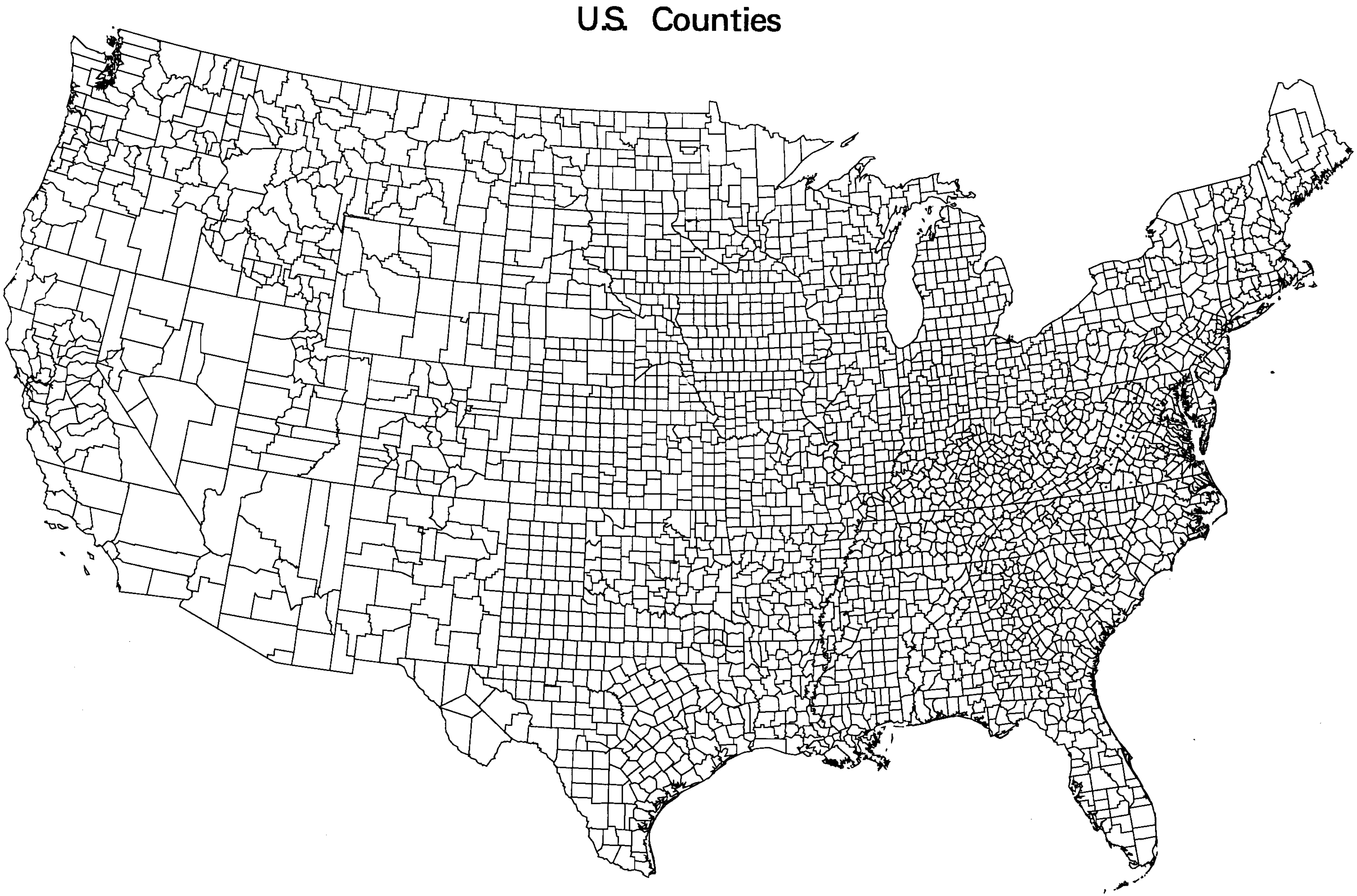
This is why most statistical almanacs do not contain from-to tables.

409 km Average Resolution

Patterns 818 km in size might be seen



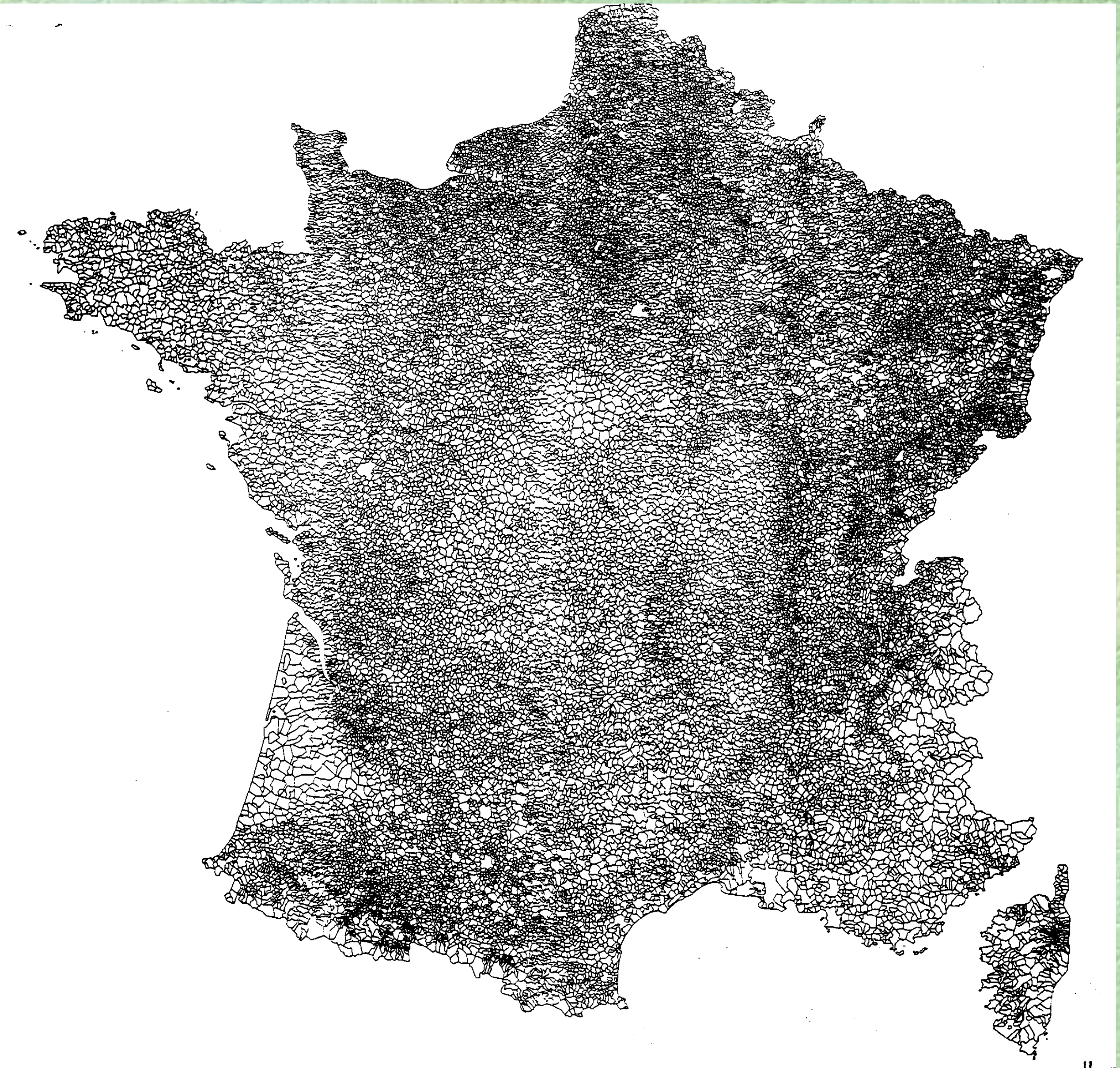
55 km Average Resolution



Think Big!

The 36,000 communes of France could yield a migration or interaction table with as many as 1,335,537,025 entries. (3 km average resolution)

France's 36,545 Communes



A table giving the interaction of everybody on earth with everyone else would be $6x10^9$ by $6x10^9$ in size, and that's only for one time interval!

But it is a very sparse table, each person having at most a few thousand connections.

In Summary

I have proceeded from very simple topographic slopes to movement models, using a variety of vector fields.

In case you wish to go further there is appended a short list of books that I have found useful.

J. Marsen, & Tromba, A., 1988, *Vector Calculus*, 3rd ed., Freeman, New York.

R. Osserman, 1968, *Two Dimensional Calculus*, Harcourt Brace, New York.

H. Schey, 1975, *Div, Grad, Curl, and all That*, **1st ed**., Norton, New York.

Thank you for your attention.