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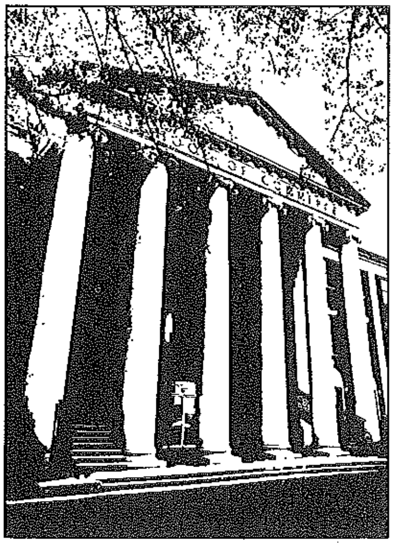

THE FORMAL AND COMPUTATIONAL RELATIONSHIP OF THE STOCHASTIC AND DETERMINISTIC MEDIAN LOCATION PROBLEMS

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Abstract

The relationship of the Stochastic Median Problem of Mirchandani to the deterministic p-Median Problem is described. It is shown that under certain conditions the size of the Stochastic Median Problem can be reduced. Implications of this relationship on solution procedures are discussed.

INTRODUCTION

Mirchandani [5] and Mirchandani and Odoni [6] have presented an interesting and important generalization of the median problem called the Stochastic Median Problem. The Stochastic Median Problem (SMP) is an extended p-Median Problem (PMP) where multiple travel time states are used. Essentially, weighted distance is minimized by the location of p facilities where the weighted distance is the sum of the weighted distances of the various states multiplied by the fraction of the time each travel time (distance) state exists. The weighted distance for each travel time state is calculated by assigning each demand to its closest facility using the travel times for that state. One state differs from another by a difference in a travel attribute value (e.g., time) of at least one link [5].

Under assumptions of homogeneity, Mirchandani [5] has proved that an optimal solution to such a problem exists at the nodes of a network. Therefore, under homogeneity conditions, an optimal solution to the SMP can be determined by finding the best p-nodal solution. In addition to the proof, Mirchandani presented an integer-linear programming formulation for identifying the best p-nodal solution. In terms of this generalized network model, Mirchandani [5] noted that some of the approaches to the PMP (notably dual based and decomposition) cannot be readily applied to the more generalized SMP, because one cannot break up the problem into p relaxed linear programming subproblems.

One purpose of this paper is to show two important properties associated with the SMP. First, we will show that the SMP can be reformulated as a p-Median problem with additional demands. This means that all p-Median solution techniques, including dual based and decomposition methods can be used to solve the SMP. In fact the application of any of the p-Median solution techniques can be accomplished with little or no modification to existing computer programs. Second, we will show that in special circumstances, the Stochastic Median Problem is equivalent to a classical median problem without additional demand nodes. Then we will compare the computational performance of two dual-based p-Median solution procedures for location problems on particular stochastic networks.

MODEL REFORMULATION

Mirchandani formulated the Stochastic Median Problem with the following notation:

n = The number of demand nodes (assume without loss of generality that the demand nodes are potential facility sites as well).

s = The number of travel time states.

$g_k(j)$ = The incident generation rate of demand node j in state k where j=1, 2, ..., n and k=1, 2, ..., s.

P_k = The probability of occurrence of state k where k=1, 2, ..., s.

$c(w_k(i,j))$ = The cost of travel (e.g., time) associated with the best route between node i and node j during state k.

$$x_{ijk} = \begin{cases} 1 & \text{if demand node } j \text{ is serviced by a facility at node } i \text{ during travel state } k. \\ 0 & \text{otherwise.} \end{cases}$$

p = The number of facilities to be located.

$$y_i = \begin{cases} 1 & \text{if node } i \text{ is selected to house a facility.} \\ 0 & \text{otherwise.} \end{cases}$$

The problem can then be formulated as:

$$\text{Min } Z = \sum_{k=1}^{s} \sum_{\substack{i=1 \\ i\neq j}}^{n} \sum_{j=1}^{n} P_k g_k(j)\, c(w_k(i,j))\, x_{ijk} \qquad (1)$$

Subject to:

$$\sum_{\substack{i=1 \\ i \neq j}}^{n} x_{ijk} + y_j = 1 \qquad \text{for each demand } j=1, 2, \ldots, n \text{ and each state } k=1, 2, \ldots, s \qquad (2)$$

$$x_{ijk} \leq y_i \qquad \text{for each } i, j, k \text{ where } k=1, 2, \ldots, s; \; i=1, 2, \ldots, n; \text{ and } j=1, 2, \ldots, n; \text{ and } i \neq j. \qquad (3)$$

$$\sum_{i=1}^{n} y_i = p \qquad (4)$$

$$x_{ijk} = 0,1 \qquad \text{for each } i, j, k \text{ where } k=1, 2, \ldots, s; \; i=1, 2, \ldots, n; \text{ and } j=1, 2, \ldots, n; \text{ and } i \neq j. \qquad (5)$$

$$y_i = 0,1 \qquad \text{for each } i=1, 2, \ldots, n \qquad (6)$$

Mirchandani did not include variables x_{iik} in the above formulation because when a unit is located at i, then the demand at i will be serviced at i regardless of the state of the network. The first constraint maintains that demand at j is assigned to a facility in each time state or that it is served by a facility at j in all time states. Essentially, the facility that j assigns to in state k can be different from the facility that j assigns to in state k'. The second constraint prevents demand j from being served by node i in state k unless a facility has been allocated to node i. The third constraint insures that exactly p facilities will be located. The last two constraints deal with integer restrictions of the decision variables y_i and x_{ijk}. The Stochastic Median Problem has been solved by a Lagrangian relaxation-subgradient optimization procedure by Weaver and Church [11]. Although Weaver and Church did make a slight change in notation in approaching the SMP, the change was made to simplify the presentation of the solution procedure. On the other hand, we will present here a general formulation which can be solved by any of the p-Median solution approaches.

In order to reformulate the Mirchandani problem, let us consider the following notation:

ℓ = the index associated with a specific pair (i,k). That is, the combinations of all pairs of demand and time states will be indexed. There are ns such pairs where n is the number of demand nodes and s is the number of travel time states.

R_ℓ = {i,k}, the pair of i,k values that are labeled as the ℓ^{th} pair.

U_i = $\{\ell | \text{such that } i \varepsilon R_\ell\}$, the set of pairs ℓ such that pair ℓ is a state of demand i.

$$x_{\ell j} = \begin{cases} 1 \text{ if node } i \text{ assigns to a unit at node } j \text{ in travel state } k, \\ \text{where } i \varepsilon R_\ell \text{ and } k \varepsilon R_\ell. \\ 0 \text{ otherwise.} \end{cases}$$

$a_\ell = P_k g_k(i)$ where $i \varepsilon R_\ell$ and $k \varepsilon R_\ell$.

$d_{\ell j} = C(w_k(i,j))$ where $i \varepsilon R_\ell$ and $k \varepsilon R_\ell$.

Using the above notation, we can make an equivalent formulation of the SMP as follows:

$$\text{Min } Z = \sum_\ell \sum_j a_\ell \, d_{\ell j} \, x_{\ell j} \tag{7}$$

S.T.

$$\sum_j x_{\ell j} = 1 \qquad \text{for each } \ell=1, 2, \ldots, L \tag{8}$$

$$x_{\ell j} \leq y_j \qquad \text{for each } \ell \text{ and } j,\ \ell=1, 2, \ldots, L \text{ and } j=1, 2, \ldots, n. \tag{9}$$

$$\sum_j y_j = p \tag{10}$$

$$x_{\ell j} = 0,1 \qquad \text{for each } \ell \text{ and } j,\ \ell=1, 2, \ldots, L \text{ and } j=1, 2, \ldots, n. \tag{11}$$

$$y_j = 0,1 \qquad \text{for each } j=1, 2, \ldots, n \tag{12}$$

This formulation is essentially that of a classical p-Median problem modified to allow for demand nodes that are not at potential facility sites (see ReVelle and Swain [9], and Cornuejols et. al. [2] as examples). The first constraint insures that for each ℓ (state k of demand i), an assignment must be made to a facility. Since the objective is to minimize weighted distance of all assignments, the assignment of a given demand in a given state will be to the closest facility (calculated on the basis of the travel times in that state). The second constraint insures that an assignment for ℓ (node $i \varepsilon R_\ell$ and state $k \varepsilon R_\ell$) cannot be made to a facility at j unless a facility has been located at j. The third constraint insures that exactly p facilities will be located. The remaining conditions represent the integer restrictions on the decision variables.

The difference between the two formulations of the SMP is that the second formulation represents each demand and state k combination by an index ℓ whereas the first formulation keeps the subscripts i and k separate. Essentially, the second formulation represents each state of a given demand i by a node. That is, there is a demand node in the new formulation for each state of each demand node in the original formulation. Since the states are represented by additional demand nodes, the explicit use of the state subscript has been eliminated in the new formulation.

Since the new formulation of the SMP is a classical p-Median problem (where there are more demand nodes than potential facility sites) the SMP can be solved by any of the wide variety of programming methods that have been developed for the PMP. This includes as well the dual based and decomposition procedures, contrary to Mirchandani's first appraisal. It should be a simple task to apply most p-Median computer routines in solving the SMP. In essence, the demand node data set needs to be expanded along with the travel times $d_{\ell j}$. Since this change is to the data set and not the program routines, it should be a straightforward task. For example, the ALLOC IV, V, and VI routines of Hillsman can be applied without any modifications, since the system easily handles such a structure [4].

SPECIAL CASES OF THE SMP

It was shown in the previous section that an s-state n-node SMP could be reformulated as a simple PMP with sn nodes and n potential facility sites (7)-(12). We will now prove that for an important special case of the SMP, (1)-(6) can be reformulated as a simple PMP with n demand nodes. The special condition for which this can be shown is when the order of closest facility sites i to a given demand node j is the same over all travel states k. This does not mean that the travel times are the same over all states k for demand j being served by the various facility sites i. It means that the relative closeness of the facility sites remains the same over all states.

Proposition. When the relative closeness of all facilities is the same for all travel states of each demand node then the SMP can be reformulated as an n-node PMP.

Proof. For the SMP formulation (1)-(6), when the relative closeness of facilities to a given demand i is the same in all travel states k, a demand node i will assign to the same facility j in all states k at the optimum (if it does not, then the relative closeness of facilities to demand node i is not the same in all states). Now the s variables x_{ijk} can be replaced by one variable $\hat{x}_{ij}$ in (1)-(6). The objective function becomes:

$$\text{MIN } Z = \sum_{k=1}^{s} \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=1}^{n} P_k g_k(j)\ c(w_k(i,j))\hat{x}_{ij} \tag{13}$$

As the outer summation does not involve decision variables it can be moved inward and (13) becomes:

$$\text{MIN } Z = \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i \neq j}}^{n} \sum_{k=1}^{s} P_k g_k(j)\ c(w_k(i,j))\hat{x}_{ij} \tag{14}$$

If we define $\hat{c}_{ij} = \sum_{k=1}^{s} P_k g_k(j)\ c(w_k(i,j))$, (15)

then we can rewrite the objective function as:

$$\text{MIN } Z = \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=1}^{n} \hat{c}_{ij}\hat{x}_{ij} \tag{16}$$

When the variables x_{ijk} are replaced in the constraints by $\hat{x}_{ij}$ the assignment constraints (2) become:

$$\sum_{\substack{i=1 \\ i \neq j}}^{n} \hat{x}_{ij} + y_i = 1 \qquad \text{for } k=1, 2, \ldots, s \quad j=1, 2, \ldots, n \tag{17}$$

the variable upper bounding constraints (3) become:

$$\hat{x}_{ij} \leq y_i \qquad \text{for } k=1, 2, \ldots, s \quad i, j=1, 2, \ldots, n, i \neq j \tag{18}$$

and the zero-one requirements (5) become:

$$\hat{x}_{ij} = 0,1 \qquad \text{for } k=1, 2, \ldots, s \quad i, j=1, 2, \ldots, n, i \neq j \tag{19}$$

Now for all constraints (17), (18) and (19) the constraints for $k \geq 2$ are duplicates and can be deleted. After simplification the formulation for the SMP for this case is as follows:

$$\text{MIN } Z = \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=1}^{n} \hat{c}_{ij}\hat{x}_{ij} \tag{20}$$

$$\sum_{\substack{i=1 \\ i \neq j}}^{n} \hat{x}_{ij} + y_i = 1 \qquad \text{for } j=1, 2, \ldots, n \tag{21}$$

$$\hat{x}_{ij} \leq y_i \qquad \text{for } i, j=1, 2, \ldots, n, \; i \neq j \tag{22}$$

$$\sum_{i=1}^{n} y_i = p \tag{23}$$

$$\hat{x}_{ij} = 0,1 \qquad \text{for } i, j=1, 2, \ldots, n, \; i \neq j \tag{24}$$

$$y_i = 0,1 \qquad \text{for } i=1, 2, \ldots, n \tag{25}$$

This is a formulation of the classical PMP with n demand nodes which are the potential facility sites. Thus, if relative closeness is the same for all states of each demand node, then each original demand i can be represented by one demand node in the new SMP formulation. For this case, the SMP can be formulated as a classical p-Median problem without additional demand nodes. Basically, the size of the SMP collapses to a simple median problem.

It is unlikely in many cases that the SMP can be represented as a p-Median problem with no additional demand nodes. In fact, it may be common that the relative closeness of facility sites over all states for a given demand may not be the same. As an example, let's assume we are dealing with a five state problem. In analyzing a particular demand node i over all travel states, we notice that in three of the states relative closeness of the facility sites is the same and that relative closeness is the same for the remaining two states (but different from the other three states). Although we cannot represent this demand i by one node in the new formulation we can represent it by two nodes ℓ' and ℓ'' (instead of 5 nodes) where:

$$a_{\ell'}d_{\ell' j} = \sum_{\ell \varepsilon C_1} a_\ell d_{\ell j} \tag{26}$$

$$a_{\ell'}d_{\ell''} = \sum_{\ell \varepsilon C_2} a_\ell d_{\ell j} \tag{27}$$

and where C_1 is a set of $\ell \varepsilon U_i$ where relative closeness is the same and C_2 is a set of $\ell \varepsilon U_i$ ($C_1 \cap C_2 = \emptyset$ and $C_1 \cup C_2 = U_i$) where relative closeness is the same. In general, we can decrease problem size whenever the relative closeness of facility sites is the same for two or more travel states of a given demand i.

Although most problems will not collapse down to a PMP with no additional demand nodes, it is now clear that additional nodes are needed only when relative closeness of facility sites differs over travel states of a given demand i. Further, any two or more states of a demand i can be represented by a node in the new formulation whenever the relative closeness of facility sites is the same over those states. The impact of this observation is that the new problem formulation can be much smaller in practice than that given in the previous section.

DUAL-BASED MEDIAN SOLUTION PROCEDURES

Dual-based mathematical programming procedures have been widely used to solve the p-Median problem. We will briefly review two such procedures and solve median location problems on two related stochastic networks to determine if the reformulated p-Median problems are more difficult to solve than standard (square) p-Median problems.

A Lagrangian dual of (7)-(12) can be formed by multiplying each assignment (8) constraint by a nonnegative multiplier and appending it to the objective function. The resulting dual can be maximized by subgradient optimization. This has been done with very good results for the p-Median problem by Cornuejols et al [2], Narula et al [8] and Mulvey and Crowder [7]. Weaver and Church [11] have extended the subgradient method (SOLD) to the SMP. In all the referenced articles [2], [7], [8] and [11] optimal solutions to the great majority of problems attempted were obtained in reasonable computer execution time without branch-and-bound coding. This may be surprising to some as the p-Median problem is NP-complete (see Cornuejols et al [2] for a discussion of the computational complexity of the p-Median problem). In spite of the fact that there is no known polynomial bound on the algorithm the subgradient procedure has been quite successful for median problems on a number of different networks. However, as will be seen below, there are data sets where the subgradient procedure does not perform so well.

Each presentation of the subgradient procedure while quite similar in spirit differs in detail. In the computational results reported below the algorithm will be executed under the conditions reported in Weaver and Church [11] with the following exceptions: (a) the target initial solution used was the p nodes with the greatest average weights; (b) the procedure was performed up to a maximum of 400 iterations; and (c) optimality was assumed verified if the dual was within 0.01 percent of the lowest primal objective value identified.

Another dual-based mathematical programming procedure, DUALOC, was developed by Erlenkotter [3] for the solution of the related fixed charge plant location problem. Erlenkotter suggested that median problems could be solved as fixed charge problems by letting each facility have the same fixed charge f and varying the fixed charge until a p-facility solution was obtained. Weighted distance in this case is just the fixed charge plant location problem objective value less pf. A major drawback to this approach is that for some values of p there may not be appropriate fixed charge. The stochastic median problem (7)-(12) can be solved in a similar fashion. If the sum of f times the facility variables is added to (7) then the objective function becomes

$$\text{MIN} \quad Z = \sum_{\ell} \sum_{j} a_\ell d_{\ell j} \, x_{\ell j} + \sum_{j} f y_i \tag{28}$$

All that remains to convert (7)-(12) into a fixed charge plant location formulation is to delete the p-facility constraint (10) then (28) subject to (8), (9), (11), and (12) is a fixed charge plant location problem with n potential facility sites and sn demand sites. The resulting fixed charge plant location problem can be solved for various values of f to obtain (hopefully) the desired p-facility solution. Several fixed charges may be required to obtain one stochastic p-Median solution. DUALOC is a very efficient procedure and does have branch-and-bound coding so that the several fixed charges required are not generally that a great computational burden. The situation where DUALOC is computationally expensive is when no fixed charge exists for the p-facility solution; then a large number of fixed charge plant location problems must be solved to establish this fact. DUALOC requires that all cost ($a_\ell d_{\ell j}$ and f in (28)) be integer valued. For further details on DUALOC the reader is referred to [3]. Van Roy and Erlenkotter [10] have made computational improvements to DUALOC which would likely result in somewhat shorter execution times than reported below. Both the subgradient procedure and DUALOC can identify and verify optimal solutions to median problems in a great many cases. However as median problems on general networks are NP-complete, it should come as no surprise that there are stochastic p-Median problems below where one or the other of these procedures does not perform as well as it did for other data sets in previously published results.

RESULTS OF COMPUTATIONAL EXPERIMENTS

To gain further computation experience solving median location problems on stochastic networks two 5-state test data sets were generated from a 25-node 42-arc data set of Berman [1]. For the first 5-state network only distance is stochastic. The state probabilities were set at 0.3, 0.25, 0.2, 0.15, 0.10 for states 1, 2, 3, 4 and 5 respectively. Distance arcs in each state were obtained by multiplying the Berman arc distance by a pseudo-random number uniformly distributed between zero and ten times the state probability and then rounding to the next greatest integer. The resulting stochastic arcs are displayed in Table 1. The distance matrices required in the mathematical programming formulations can be obtained by any shortest path algorithm for each state separately. The first test data set generated consisted of the stochastic arc distance in Table 1 and the original Berman population weights shown in Table 2. Another 5-state test data set was generated utilizing the stochastic arcs in Table 1 and stochastic node weights. Stochastic node weights were determined by multiplying the Berman node weight by a pseudo-random number uniformly distributed between zero and ten times the state probability then normalizing the node weights to sum to one in each state. The resulting node weights are reported in Table 2. The state probabilities have a maximum of two nonzero digits passed the decimal point and the node weights have a maximum of three nonzero digits passed the decimal. The multiplication required to obtain integer cost results in an integer weighted distance which is 10^5 times the weighted distance of the data sets.

TABLE 1

25-NODE NETWORK ARC DISTANCES

Arc	Between Nodes		Distance					
			Berman	State 1	State 2	State 3	State 4	State 5
1	1	5	5	2	8	5	3	3
2	5	6	5	14	11	6	2	4
3	1	2	4	7	2	6	3	2
4	4	5	3	2	7	5	3	2
5	5	7	5	12	12	1	2	4
6	6	7	3	8	4	6	1	3
7	2	4	4	12	3	8	4	4
8	4	7	5	8	5	1	2	4
9	7	12	9	23	8	3	12	8
10	2	3	3	8	2	6	2	2
11	3	4	4	6	6	7	1	3
12	4	9	7	15	7	8	9	1
13	4	8	5	6	13	10	7	5
14	7	8	3	7	2	5	4	1
15	7	11	8	5	5	16	8	3
16	11	12	2	5	1	4	1	2
17	12	16	4	3	4	2	6	2
18	12	15	4	1	10	5	4	2
19	3	9	4	3	2	1	3	1
20	8	9	6	13	14	11	2	4
21	8	11	7	13	3	12	6	1
22	24	25	8	16	1	2	10	7
23	15	16	4	4	5	1	4	1
24	9	10	6	6	14	12	1	4
25	8	10	6	12	15	12	1	5
26	8	13	7	1	7	11	3	1
27	11	13	3	7	4	1	2	3
28	10	13	6	2	11	5	8	5
29	10	14	3	8	7	6	3	2
30	13	14	7	9	5	2	3	1
31	13	19	4	11	3	6	5	2
32	16	17	4	5	5	8	6	4
33	14	19	7	14	16	14	4	6
34	17	19	3	2	3	6	2	2
35	14	22	4	11	4	6	5	1
36	14	21	2	5	2	2	2	1
37	19	20	3	6	1	5	4	3
38	17	18	3	4	6	4	1	1
39	20	21	2	3	5	4	3	1
40	18	20	3	7	7	2	3	3
41	22	23	3	3	4	5	2	2
42	23	24	3	7	4	5	3	3
STATE PROBABILITY			1.0	0.3	0.25	0.2	0.15	0.1

TABLE 2

25-NODE NETWORK
NODE WEIGHTS

Node	Berman	State 1	State 2	State 3	State 4	State 5
1	.050	.058	.068	.016	.077	.022
2	.082	.046	.084	.058	.068	.011
3	.023	.020	.038	.007	.009	.022
4	.032	.025	.033	.010	.007	.001
5	.023	.013	.036	.011	.025	.038
6	.003	.004	.004	.001	.007	.005
7	.007	.002	.004	.000	.016	.004
8	.061	.109	.067	.056	.072	.076
9	.013	.011	.013	.016	.022	.020
10	.052	.059	.044	.090	.043	.019
11	.013	.014	.005	.014	.005	.008
12	.052	.054	.055	.036	.041	.067
13	.005	.003	.005	.008	.007	.001
14	.059	.061	.055	.006	.063	.079
15	.017	.008	.016	.023	.017	.012
16	.022	.011	.037	.029	.051	.025
17	.038	.070	.039	.008	.088	.032
18	.104	.165	.101	.171	.129	.160
19	.070	.060	.086	.073	.012	.028
20	.036	.007	.064	.054	.084	.039
21	.018	.015	.027	.023	.031	.015
22	.073	.072	.089	.031	.006	.115
23	.011	.021	.017	.016	.012	.010
24	.134	.092	.009	.238	.105	.190
25	.002	.000	.004	.005	.003	.001

The one through ten median SOLD solutions for the 25-node 5-state network with the Berman node weights are reported in Table 3. For each SMP the execution time, the number of iterations required, weighted distance and the location configuration are noted. For the 5-median and 6-median problems where solutions were not verified as optimal by the subgradient procedure the Lagrangian bound is noted. The SOLD procedure identified and verified an optimal solution for eight out of the ten cases and determined a bound which shows that the other 2 solutions are within one percent of optimality.

The computational result for DUALOC solution of the 25-node 5-state median data set with Berman node weights are reported in Table 4. DUALOC obtained and verified an optimal solution to all facility problems between one and ten. For five of the ten SMPs more than one fixed charge was required; but even so the total time required for each number of facilities was generally less than that required by the subgradient procedure. Branching was required for several of the fixed charge problems. DUALOC verified as optimal the 5-median and 6-median solution identified by the subgradient procedure. Both procedures performed reasonably well on this data set.

The performance for the SOLD procedure noted in Table 3 is in general accordance with that reported in Weaver and Church [11] for another data set with stochastic distances and static population weights. The computational effort required to solve a stochastic median problem with DUALOC is somewhat dependent upon the fixed charge estimate used, as can be seen for the 4-median and 7-median results in Table 4 where different fixed charges result in significantly different execution times and branching requirements.

The subgradient procedure and DUALOC were less successful in solving median location on the 25-node 5-state network with stochastic distances and

TABLE 3

25-NODE NETWORK WITH
STOCHASTIC DISTANCE
SUBGRADIENT SOLUTIONS

P	Execution Time (a)	Numbers of Iterations	Weighted Distance(b)(c)	Facilities Located at Nodes
1	0.85	27	1081583	13 [illegible]
2	2.50	108	778600	13,24
3	0.66	28	504583	4,19,24
4	2.47	107	379880	2,13,18,24
5	9.06	400	301802 (299572)	2,8,14,17,24
6	9.57	400	245238 (242814)	2,10,12,18,22,24
7	1.74	68	191193	2,10,12,18,19,22, 24
8	1.75	63	146364	2,8,12,14,18,19, 22,24
9	3.60	127	114772	2,5,10,12,14,18, 19,22,24
10	2.66	89	91681	2,5,8,10,12,14,18, 19,22,24

(a) Execution time in seconds on a UNIVAC 1100/61 computer.

(b) Weighted distance is multiplied by 10^5 to facilitate comparison with DUALOC solutions.

(c) If optimality is not verified within 0.01 percent, the Lagrangian bound is noted under the lowest weighted distance identified in parenthesis.

TABLE 4

25-NODE NETWORK WITH
STOCHASTIC DISTANCE
DUALOC SOLUTIONS

P	Fixed Charge	Execution Time (a)	Number of Iterations	Branching Required?	Objective Value	Weighted Distance(b)
1	800000	0.69	1	No	1881583	1081583
2	300000	1.12	7	No	1378600	778600
3	250000	1.04	5	No	1254583	504583
	150000	0.96	3	No	954583	504583
4	100000	1.01	5	No	779880	379880
	80000	3.20	65	Yes	699880	379880
5	70000	3.02	54	Yes	651802	301802
	60000	3.09	55	Yes	601802	301802
6	55000	2.46	45	Yes	575238	245238
7	50000	1.87	21	No	541193	191193
	45000	2.33	20	Yes	506193	191193
8	40000	1.17	10	No	466364	146364
9	30000	0.86	12	Yes	384772	114772
	25000	0.84	11	Yes	339772	114772
10	20000	0.58	3	No	291681	91681

(a) Execution time in seconds on a UNIVAC 1100/61 computer.

(b) Weighted distance is multiplied by 10^5 because DUALOC requires integer cost.

stochastic population weights. In particular, the subgradient method did not verify optimal solution to seven of the ten median location problems attempted. Table 5 records the results when SMPs on this 25-node 5-state network were solving using SOLD. However the subgradient procedure did identify nine of the ten optimal solutions (as verified by other methods). In all cases the Lagrangian bound was within 4 percent of the best primal solution identified. As a heuristic the subgradient procedure without branch-and-bound coding performed reasonably well on this data set, however if verified optimal solutions are required the results displayed in Table 5 indicate that such a procedure is not always a reliable method of obtaining them.

Partial results of DUALOC solutions of stochastic median problems on the 5-state 25-node network with random distance and random weight are shown in Table 6. The results are partial because well over twenty fixed charges were attempted to obtain a 2-median solution. The solutions which resulted from most of these futile attempts are not recorded, but the 1-median and 3-median solutions which indicate there is not a fixed charge for the 2-median problem are. DUALOC which has branch-and-bound coding was able to identify and verify optimal solutions to all problems attempted except the 2-median problem. DUALOC verified as optimal the 5, 7, 8, 9, and 10 median solutions obtained by the SOLD method. It would appear that neither DUALOC or the subgradient procedure in their current form is a completely satisfactory solution procedure for the SMP with random distance and random node weights. However enough success has been demonstrated to indicate that for SMPs dual-based mathematical programming procedures are often able to obtain verified optimal location configurations.

TABLE 5

25-NODE NETWORK WITH
STOCHASTIC DISTANCE AND STOCHASTIC WEIGHTS
SUBGRADIENT SOLUTIONS

P	Execution Time(a)	Number of Iterations	Weighted Distance(b)	Lagrangian Bound(c)	Facilities Located at Nodes
1	1.25	41	1038275		13
2(d)	9.20	400	786126	773416	13,24
3	5.20	221	511958		4,19,24
4	8.43	370	363594		2,13,18,24
5	9.44	400	297639	288494	2,12,13,18,24
6	9.84	400	240694	232496	2,8,12,14,18,24
7	10.43	400	183916	182998	2,8,12,14,18,19, 24
8	11.10	400	145301	140187	2,8,12,14,18,19, 22,24
9	11.65	400	114741	110368	2,5,8,12,14,18, 19,22,24
10	12.26	400	88590	85037	2,5,8,10,12,14, 18,19,22,24

(a) Execution time in seconds on a UNIVAC 1100/61 computer.

(b) Weighted distance is multiplied by 10^5 to facilitate comparison with DUALOC solutions.

(c) The Lagrangian bound is noted only if optimality is not verified.

(d) Solution verified as optimal by complete enumeration in 0.96 seconds.

TABLE 6

25-NODE NETWORK WITH
STOCHASTIC DISTANCE AND STOCHASTIC WEIGHTS
DUALOC SOLUTIONS

P	Fixed Charge	Execution Time (a)	Number of Iterations	Branching Required?	Objective Value	Weighted Distance(b)
1	263159	2.39	55	Yes	1301434	1038275
3	263158	1.87	49	Yes	1301432	511958
4	100000	0.48	1	No	763594	363594
	80000	0.99	4	No	683594	363594
5	60000	2.15	50	Yes	597639	297639
6(c)	56666	3.40	78	Yes	578744	238748
7	53333	1.75	32	Yes	557247	183916
	50000	1.26	13	No	533916	183916
	40000	1.60	10	No	463916	183916
8	36666	1.14	9	No	438629	145301
	33333	1.71	13	No	411965	145301
9	30000	2.34	22	No	354741	114741
10	20000	0.52	2	No	288590	88590

(a) Execution time in seconds on a UNIVAC 1100/61 computer.

(b) Weighted distance is multiplied by 10^5 because DUALOC requires integer cost.

(c) Optimal 6-median solution is located at nodes 2,8,14,18,19,24.

SUMMARY

We have presented a reformulation of the generalized median problem called the Stochastic p-Median Problem. Although there is only a subtle difference between this reformulation and the one originally given by Mirchandani, the new formulation can be easily solved by existing p-Median solution techniques including those originally ruled out by Mirchandani for solving the SMP. Under certain conditions of relative closeness of facility sites in travel states, we have made the observation that the SMP can collapse into a simple median problem. In many cases, it is also possible to resolve the SMP with far fewer variables and constraints than contained in the complete SMP formulation and reformulated SMP.

We have compared the performance of two dual-based solution procedures, DUALOC and the SOLD on a network with stochastic distances. DUALOC obtained optimal solutions for all median location problems attempted on the 25-node 5-state network with stochastic distances and static node weights. For the 25-node 5-state network, where both the node weights and distances were random, optimal solutions to all problems attempted except the 2-median problem were obtained with DUALOC. The subgradient procedure identified optimal solutions to nine of the ten problems on this network but verified these solutions as optimal for only three of the nine cases. The computer execution time for all problems attempted are quite reasonable. For the problems where both DUALOC and the subgradient procedure obtained verified optimal solution there does not appear to be any significant difference in computer execution time required. Where the subgradient procedure does not verify a solution as optimal or when no fixed charge exists for a given median problem then differences in computer requirements of the two methods are apparent.

Improvements are needed in current computer codes to increase the likelihood of obtaining verified optimal solutions, however, even with enhanced coding there may be particular problems which can not be solved optimally in a reasonable amount of execution time as the stochastic median location problem is NP-complete. It should be noted that for all the problems attempted in this paper an optimal solution was identified and verified by some method.

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